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A NEW APPROACH TO THE ESTIMATION OF THE DISPERSION COMPONENTS

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SUMMARY: A new method to calculate simultaneosly both the unknown X and the components of the dispersion (results of measurement) is developed. An improved iterative algorihm is used.

1. INTRODUCTION

The problem of determining dispersion parameters is not new. Among the first authors having occupied themselves with this problem was Jozef Böhm (1964). He inserted the series of squares of deviations into the equations of correction as free terms. Proceeding from them he determined the dispersion components as unknown values by intermediate leveling. The present author in his paper – Vračarić (1978) – amended prof. Böhm's idea in that he determined in proper way the weights of the square deviations which take part in the estimation of dispersion components.

The second problem appears when determinition of values of unknown parameters and components of dispersion from series of measurements is needed. A great namber of experts considered this problem: Kllefe, Rao, Koch. Method of simultanious determination of unknown values and components of dispersion is known as MINQUE method in literature. Basic criterion for estimation of components of dispersion is to obtain minimal track of covariance matrix. New method of determining values of unknown parameters and components of dispersion giving less track then MINQUE method will be shown in the the present paper.

The main deficiency of works of the above mentioned authors was that the improvements of the corrections based on the indirect adjustments were not used to estimate the dispersion components. Therefore, to improve the method, a simultaneous determination of both the dispersions components σ_i^2 and unknown variables X_i is the principal topic of this paper. An improved iterative algorithm is utilized.

2. THEORY

Let us suppose that the measurement produced the following set of n results

$$\mathbf{h}^T = [\,h_1 \quad h_2 \quad ... \quad h_n\,] \quad . \tag{1}$$

The unknown variables X_i and the correspondent dispersion components σ_i^2 are related by the well-known Gauss-Markov model

$$\begin{aligned} \mathbf{AX} &= \mathbf{E}(h) \\ \mathbf{D}(h) &= \sigma_1^2 Q_1 + \sigma_2^2 Q_2 + \ ... + \sigma_r^2 Q_r \quad . \end{aligned} \tag{2}$$

The correspondent uncorrelated errors ε_i produce a vector $\mathbf{V}$:

$$\mathbf{V} = \varepsilon_1 \mathbf{T}_1 + \varepsilon_2 \mathbf{T}_2 + \ ... \ + \varepsilon_r \mathbf{T}_r \tag{3}$$

($\mathbf{T_i}$ are **nxn** matrices).

Thus the dispersion $\mathbf{D}(h)$ is

$$\mathbf{D}(h) = \sigma_1^2 \mathbf{T}_1^T \mathbf{T}_1 + \sigma_2^2 \mathbf{T}_2^T \mathbf{T}_2 + ... + \sigma_r^2 \mathbf{T}_r^T \mathbf{T}_r \ . \tag{4}$$

It is rather convenient to write:

$$\mathbf{T}_i^T \mathbf{T}_i = \mathbf{Q}_i \tag{5}$$

and thus relation (4) could be written as:

$$\mathbf{D}(h) = \sigma_1^2 \ \mathbf{Q}_1 + \sigma_2^2 \mathbf{Q}_2 + ... + \sigma_r^2 \mathbf{Q}_r \quad . \tag{6}$$

The correlation matrix $\mathbf{Q}_h$ is

$$\mathbf{Q}_h = \frac{\mathbf{D}(h)}{\sigma^2} \quad . \tag{7}$$

It is a well-known fact that the correspondent covariant matrix $\mathbf{D}$(V) could be used instead.

The main difficulty is that the value of $\mathbf{D}$(h) is necessary to determine the vector $\mathbf{X}$:

$$\mathbf{X} = (\mathbf{A}^T \mathbf{Q}_h^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Q}_h^{-1} \mathbf{h} \tag{8}$$

and also, the value is necessary to determine $\mathbf{D}(h)$. Evidently, an iterative algorithm has to be used.

The first iteration is as follows

$$(\sigma_i^2)_h = 1 \tag{9}$$

and therefore:

$$(\mathbf{Q}_h)_1 = \mathbf{Q}_1 + \mathbf{Q}_2 + ... + \mathbf{Q}_r \tag{10}$$

and the first iterations of $\mathbf{X}$ and $\mathbf{V}$ are:

$$(\mathbf{X})_1 = (\mathbf{A}^T (\mathbf{Q}_h)_1^{-1} \mathbf{A})^{-1} \mathbf{A}^T (\mathbf{Q}_h)_1^{-1} \ \mathbf{h} \tag{11}$$

$$(\mathbf{V})_1 = \mathbf{A}(\mathbf{X})_1 - \mathbf{h} = (A(\mathbf{A}^T (\mathbf{Q}_h)_1^{-1} \mathbf{A})^{-1} \mathbf{A}^T (\mathbf{Q}_h)_1^{-1} - \mathbf{I}) \ \mathbf{h} \tag{12}$$

($\mathbf{I}$ is a unity **nxn** matrix).

Furthermore

$$(\mathbf{V})_1 = (\mathbf{R})_1 \mathbf{h} \tag{13}$$

where a simetrical **nxn** matrix $\mathbf{R}$ is:

$$\mathbf{R} = A(\mathbf{A}^T (\mathbf{Q}_h)_1^{-1} \mathbf{A})^{-1} \mathbf{A}^T (\mathbf{Q}_h)_1^{-1} - \mathbf{I} \quad . \tag{14}$$

It is known that the dispersion $\mathbf{D}$(h) is equal to the expectation $\mathbf{E}(V^2)$; therefore

$$\mathbf{E}(V^2) = \mathbf{D}(h) \ = \sigma^2 \mathbf{Q}_v \ = \sigma^2 \mathbf{R}^T \mathbf{Q}_l \mathbf{R} = \sigma_1^2 \mathbf{Q}_1 + \sigma_2^2 \mathbf{Q}_2 + ... + \sigma_r^2 \mathbf{Q}_r \quad . \tag{15}$$

To calculate the expectation $\mathbf{E}(V^2)$ a great series of measurements should be performed

$$D(h) \ = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} V_i^2}{n} \quad . \tag{16}$$

Practically, only the correspondent estimation could be found

$$\hat{D}(h) = \frac{[V^2]}{n} \quad . \tag{17}$$

It predominant cases, n=1, and

$$\hat{D}(h) = V^2 \quad . \tag{18}$$

Thus, it is evident that some corrections must be included:

$$\overline{\mathbf{V}} = \sigma_1^2 \mathbf{Q}_1 + \sigma_2^2 \mathbf{Q}_2 + ... + \sigma_r^2 \mathbf{Q}_r - \mathbf{V}^2 \tag{19}$$

or

$$\overline{\mathbf{V}} = \mathbf{B}\sigma - \mathbf{V}^2 \tag{20}$$

and

$$\mathbf{B} = \begin{bmatrix} Q_{1_{11}} & Q_{2_{11}} & ... & Q_{r_{11}} \\ Q_{1_{21}} & Q_{2_{22}} & ... & Q_{r_{22}} \\ ... & ... & ... & ... \\ Q_{1_{nn}} & Q_{2_{nn}} & ... & Q_{r_{nn}} \end{bmatrix} \quad . \tag{21}$$

Finally:

$$\sigma = (\mathbf{B}^T \mathbf{Q}_{V^2}^{-1} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{Q}_{V^2}^{-1} \mathbf{V}^2 \quad . \tag{22}$$

3. RESULTS

To calculate σ it is necessary to determine the correlation matrix $\mathbf{Q}_{V^2}$. It was shown (Mihailović K., 1992), that the covariant matrix of two square forms:

$$\mathbf{C}(\mathbf{h}^T \mathbf{A}_i \mathbf{h}, \mathbf{h}^T \mathbf{A}_j \mathbf{h}) = 2tr(\mathbf{A}_i \mathbf{Q}_h \mathbf{A}_j \mathbf{Q}_h) + 4\mu_h^T \mathbf{A}_i \mathbf{Q}_l \mathbf{A}_j \mu_h \quad . \tag{23}$$

In the present case , the covariance of $\mathbf{V}$ is determined; therefore $\mu_V = 0$ and:

$$\mathbf{C}(\mathbf{h}^T \mathbf{A}_i \mathbf{h}, \mathbf{h}^T \mathbf{A}_j \mathbf{h}) = 2tr(\mathbf{A}_i \mathbf{Q}_h \mathbf{A}_j \mathbf{Q}_h) \quad . \tag{24}$$

If **nxn** matrices $\mathbf{A}_i$ have been chosen so that in any matrix $\mathbf{A}_i$

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} ; \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} ; \quad \mathbf{A}_n = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix} \tag{25}$$

the consequence is:

$$\mathbf{V}^T \mathbf{A}_i \mathbf{V} = \mathbf{V}_i^2 \qquad i = 1, \ 2, \ ... \ n \quad . \tag{26}$$

The correlation matrix of adjustments is written as:

$$\mathbf{Q}_V = \begin{bmatrix} Q_{11} & Q_{12} & ... & Q_{1n} \\ Q_{21} & Q_{21} & ... & Q_{2n} \\ ... & ... & ... & ... \\ Q_{n1} & Q_{n2} & ... & Q_{nn} \end{bmatrix} \quad . \tag{27}$$

Then the covariance of square errors is

$$\mathbf{C}(\mathbf{V}^T\mathbf{A}_i\mathbf{V},\ \mathbf{V}^T\mathbf{A}_j\mathbf{V}) = \mathbf{C}(\mathbf{V}_i^2,\ \mathbf{V}_j^2) = 2tr(\mathbf{A}_i\mathbf{Q}_V\mathbf{A}_j\mathbf{Q}_V) = 2\mathbf{Q}_{ij}^2 \quad . \tag{28}$$

Therefore

$$\mathbf{Q}_{V^2} = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & ... & Q_{1n}^2 \\ Q_{21}^2 & Q_{22}^2 & ... & Q_{2n}^2 \\ ... & ... & ... & ... \\ Q_{n1}^2 & Q_{n2}^2 & ... & Q_{nn}^2 \end{bmatrix} \quad . \tag{29}$$

As $\mathbf{Q}_{V^2}$ is determined, one can calculate σ [Eq. 22], and thus the first iteration step is completed. The iteration procedure further continues to obtain the satisfactory accuracy. This result has to be the best unbiased estimation.

To ilustrate the sugested new method of estimation of dispersion components, a numerical example, shown in literature (Koch 1988., pages: 275 and 276), has been done.

Succession of measurements is given, for instance altitude differences determined by trigonometric heighting by two different instruments. Declared accuracy of measuring for the first instrument is 1 cm and for second 2 cm. Determin dispersion components and track of covariance matrix using the MINQE method and using new method suggested in his work

$$\mathbf{A}= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} ;$$

$$\mathbf{h}= \begin{bmatrix} 1.25 \\ 1.26 \\ 1.24 \\ 1.22 \\ 1.27 \end{bmatrix} ;$$

$$\mathbf{T1}= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ;$$

$$\mathbf{T2}= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ;$$

$$\mathbf{a}= 10^{-4} \begin{bmatrix} 1 \\ 4 \end{bmatrix} .$$

The course of work applying MINQUE method

$$\mathbf{H1} = a(1)\mathbf{T1} \ ;$$
$$\mathbf{H2} = a(2)\mathbf{T2} \ ;$$
$$\mathbf{H} = \mathbf{H1} + \mathbf{H2} \ ;$$
$$\mathbf{F} = \mathbf{H}^{-1} - \mathbf{H}^{-1}\mathbf{A}\,(\mathbf{A}^T\,\mathbf{H}^{-1}\,\mathbf{A})^{-1}\,\mathbf{A}^T\,\mathbf{H}^{-1} \ ;$$
$$S_{(1,1)} = tr(\mathbf{F}\mathbf{H}_1\mathbf{F}\mathbf{H}_1) \ ;$$
$$S_{(1,2)} = tr(\mathbf{F}\,\mathbf{H}_1\mathbf{F}\,\mathbf{H}_2) \ ;$$
$$S_{(2,1)} = S_{(1,2)} \ ;$$
$$S_{(2,2)} = tr(\mathbf{F}\,\mathbf{H}_2\mathbf{F}\,\mathbf{H}_2) \ ;$$
$$q_{(1)} = tr(\mathbf{h}^T\mathbf{F}\mathbf{H}_1\mathbf{F}\mathbf{h}) \ ;$$
$$q_{(2)} = tr(\mathbf{h}^T\mathbf{F}\mathbf{H}_2\mathbf{F}\,\mathbf{h}) \ ;$$
$$\mathbf{d} = \mathbf{S}^{-1}\,\mathbf{q} \ ;$$
$$al(1) = a(1)\ d(1) \ ;$$
$$al(2) = a(2)\ d(2) \ .$$

Now, the values **a** are changed by **a1** and iterations are repeated.

After few iterations we get:

$$\sigma^2 = 10^{-4} \begin{bmatrix} 0.9614 \\ 6.7493 \end{bmatrix} \ ;$$

$$tr(\mathbf{H}) = 0.00164 \ .$$

The course of work applying new method

$$\mathbf{H1} = a(1)\ \mathbf{T1} \ ;$$
$$\mathbf{H2} = a(2)\ \mathbf{T2} \ ;$$
$$\mathbf{H} = \mathbf{H1} + \mathbf{H2}$$
$$\mathbf{E} = (5x5) \quad \text{unity matrix} \ ;$$
$$\mathbf{R} = \mathbf{E} - \mathbf{A}(\mathbf{A}^T\mathbf{H}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{H}^{-1} \ ;$$
$$\mathbf{B} = [\text{diag}\mathbf{T1}\ \ \text{diag}\mathbf{T2}] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \ ;$$
$$\mathbf{X} = (\mathbf{A}^T\mathbf{H}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{H}^{-1}\mathbf{h} \ ;$$
$$\mathbf{V} = \mathbf{R}\ \mathbf{h} \ ;$$
$$\mathbf{Q}_v = \mathbf{R}^T\mathbf{H}\mathbf{R} \ ;$$
$$\mathbf{v1}_{(i)} = \mathbf{V}_{(i)}^2 \ ;$$
$$\mathbf{Q}_{\mathbf{v1}_{(i,j)}} = \mathbf{Q}_{\mathbf{V}_{(i,j)}}^2 \ ;$$
$$\sigma^2 = (\mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{v1}}^{-1}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{v1}}^{-1}\mathbf{v1} \ .$$

The obtained values of σ^2 are changed as **a** and iteration is repeated.

After THREE iterations we get:

$$\sigma^2 = 10^{-4} \begin{bmatrix} 0.66770 \\ 6.46885 \end{bmatrix} \ ;$$

$$\mathbf{X} = 1.24968 \ ;$$

$$tr(\mathbf{H}) = 0.00149\ .$$

The basic criterion of process validity is to get minimal trace of matric **H**.

It is obvious that the application of the new method gives less trace of matrix H than the application of MINQUE method.

Number of iterations, needed to obtain final values of dispersion components can be said not to be exactly determined in the literature. The number of iterations can be discussed only from experience in using a particular method.

The number of iterations depends on the number of important digits showing the final result. The greater is number of important digits in the results the greater is the number of iterations needed to obtain final result. The number of important digits of final solution must be adjusted with the number of important digits the solution must be adjusted with the number of important digits and the number of decimal places of measurement result.

Measurement results are not obtained with absolute accuracy, but with particular errors, which are usually accidental errors of measurement, the systematic errors of measurement and errors of rounding off numbers. Accordingly a number of important digits figures in the results of measurement. The estimation of measurement results obtained after leveling will have the same number of important digits as the measurement results, possibly one more. The correction V obtained as differences between the estimated velues of measurements results and the real measured velues as a rule have little velues, mostly consisting of one important digit and rarely of two. As dispersion components are determined from the corrections V, there is no sense in determing them with greater numbers of important digits. In such a case, as experience, shows it is usually enough to do two to four iterations. If, from corrections V having one or two important digits, dispersion components with great number of important digits are determined, it is clear that the number of iterations will be much greater.

In the numerical example previously shown, dispersion components with more or fewer important digits were determined. At that the numbers of iterations were obtained, depending on the number of important digits. For instance:

When using two important digits it is obtained:

$$\sigma^2 = 10^{-4} \begin{bmatrix} 0.96 \\ 6.75 \end{bmatrix}\ ;$$

$$tr(\mathbf{H}) = 0.0016$$

after five iterations by MINQUE method

$$\sigma^2 = 10^{-4} \begin{bmatrix} 0.67 \\ 6.47 \end{bmatrix}\ ;$$

$$\mathbf{X} = 1.250\ ;$$

$$tr(\mathbf{H}) = 0.0015$$

after three iterations by method suggested in this work.

When using three important digits it is obtained

$$\sigma^2 = 10^{-4} \begin{bmatrix} 0.961 \\ 6.751 \end{bmatrix}\ ;$$

$$tr(\mathbf{H}) = 0.00164$$

after five iterations by MINQUE method

$$\sigma^2 = 10^{-4} \begin{bmatrix} 0.668 \\ 6.469 \end{bmatrix}\ ;$$

$$\mathbf{X} = 1.250\ ;$$

$$tr(\mathbf{H}) = 0.00149$$

after three iterations by method suggested in this work.

When using five important digits it is obtained:

$$\sigma^2 = 10^{-4} \begin{bmatrix} 0.96116 \\ 6.75115 \end{bmatrix}\ ;$$

$$tr(\mathbf{H}) = 0.00163858$$

after eight iterations by MINQUE method and:

$$\sigma^2 = 10^{-4} \begin{bmatrix} 0.66770 \\ 6.46884 \end{bmatrix}\ ;$$

$$\mathbf{X} = 1.2497\ ;$$

$$tr(\mathbf{H}) = 0.0014941$$

after four iterations by method suggested in this work.

REFERENCES

Böhm, J.: 1964, *Vyrovnavac počet*, SNTL, Praha.

Kleffe, J.: 1980, *Math Operationsforsch. Statist. Ser. Statistics*, 11, 563.

Koch, K.R.: 1988, Parameter Estimation and Hypothesis Testing in Linear Models, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

Mihailović, K.: 1992, Izravnanje geodetskih mreža, Naučna knjiga, Beograd.

Rao, C.R.: 1979, MINQUE - theory and it relation to ML and MML estimation of vacinace coponentrs, *Sankhya*, **B41**, 138.

Vračarić, K.: 1978, PhD thesis, Faculty of Civil Engineering Univesity of Belgrade, Belgrade.

Vračarić, K.: 1987, *Geodetski list* 7-9, 243.

НОВИ ПОСТУПАК ОЦЕНЕ КОМПОНЕНТИ ДИСПЕРЗИЈЕ

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У раду се даје теоријско образложење и предлаже нови поступак оцене компоненти дисперзије.