Bull. Astron. Belgrade № 154 (l996), 1 – 8 **UDC 52-423**
Original scientific paper

ON THE TWO-BODY PROBLEM IN MANEFF-TYPE FIELDS

C. Stoica[1] and V. Mioc[2]

[1] *Institute for Gravitation and Space Sciences, Laboratory for Gravitation, Bd. Ana Ipatescu 21, 71111 Bucharest, Romania*

[2] *Astronomical Institute of the Romanian Academy, Astronomical Observatory Cluj-Napoca, Str. Ciresilor 19, 3400 Cluj-Napoca, Romania*

(Received: May 21, 1996)

SUMMARY: The two-body problem in Maneff-type fields (characterized by potentials of the form $A/r + B/r^2$; r=distance between particles, A, B=constants) constitutes a good model for various astronomical (and also physical, astrophysical, mechanical) problems. The relative motion in such fields is being studied from analytic, geometric, and physical standpoints. An investigation with qualitative character, based on the study of the phase portraits, is being performed for the whole allowed interplay among field parameters, angular momentum, and total energy. Treating separately the nonradial one, each allowed trajectory in the phase plane is interpreted in terms of physical motion. All possible scenarios, eleven in the nonradial case, nine in the radial one, are identified. They illustrate three general behaviours (motion ending in collision, periodic/quasiperiodic orbits, escape trajectories) and two special ones (equilibria and orbits tending to equilibria.).

1. INTRODUCTION

Force fields with quasihomogeneous potential functions of the form $A/r + B/r^2$ (r= distance between particles; A, B=real constants) have been considered as far as back as three centuries ago. Newton was the first to take into account such a model in his attempts to explain the Moon's perigee motion, then Clairaut for the same purpose; Einstein also used it as a possible alternative to relativity) to compute Mercury's perihelion advance (see Diacu *et al.* 1995).

One knows that the perihelion advance of Mercury and of the other inner planets cannot be fully explained within the framework of the classical Newtonian law, even resorting to perturbation theory. The many pre- and post-relativistic gravitational laws usually answered this question, but failed to explain other issues (as the secular motion of the Moon's perigee). As regards the general relativity, it succeeded in explaining well such phenomena, from both quantitative and qualitative standpoint. Unfortunately, this powerful theory, which answered a lot of important questions in physics and astronomy, is not of much help for celestial mechanics; all attempts to formulate a meaningful relativistic n-body problem have failed to provide valuable results.

The problem is therefore to find a model able to respond to the theoretical needs of celestial mechanics, to keep the simplicity and the advantages of the Newtonian one, and also to bring the necessary corrections such that orbits coming close to collisions match theory with observations; in other words, a model able to maintain the dynamical astronomy within the framework of classical mechanics, offering at the same time equally good justifications of observed phenomena as the relativity.

Such a model is that based on the above $A/r + B/r^2$ potential law. Considerations of physical nature guided Maneff (1924, 1925, 1930a,b) to propose a similar gravitation model (with A, B positive constants, suitably concretized). Fallen into oblivion for half a century, then brought forward by Hagihara (1975) as providing the same good theoretical approximations as the relativity, Maneff's law was recently reconsidered in a series of studies having a departure point Dicu's (1993) researches. For the two-body problem with this law, Mioc and Stoica (1995a,b,c) obtained the general solution of the generalized in the velocity plane, while Diacu *et al.* (1995) found the analytic solution and the local flow near collision. Maneff's field was also used by Diacu (1993) to study the isosceles three-body problem, and by Ureche (1995) to an astrophysical problem: the free-fall collapse of a homogeneous sphere.

We call *Maneff-type field* a force field characterized by a potential function of the above form in which A and B admit *any* real values. Such generalizations, but only for *positive* A and B, were already tackled: Lacomba *et al.* (1991) studied it for negative total energy; Casasayas *et al.* (1993) computed the Melnikov integral associated with the nonhyperbolic equilibria; Diacu (1996 pointed out the special place of this potential among all quasihomogeneous potentials within the framework of the three-body problem; Delgado *et al.* (1996) provided the complete analytic, geometric and physical description of the two-body problem.

One might say (and physicists do it often): to find the motion in the two-body problem associated to the $A/r + B/r^2$ potential is an old and well-known exercise (e.g. Goldstein 1980, p. 123, Problem 14). Leaving aside the fact that Goldstein's statement is incorrect, the above quoted results, especially those of Delgado *et al.* (1996), show how complex is the problem in reality. Moreover, assigning to A and B concrete expressions, various physical and astronomical situations can be modelled. The motion in certain post-Newtonian fields, nonrelativistic (obviously, Maneff's one included) or relativistic (e.g. Fock's one (see Mioc, 1994), or that described by the Reissner-Nordström metric, truncating the negligible terms), is such a situation. The motion in the photogravitational field of a luminous source (whose gravitational action is not necessarily Newtonian) also joins this model; if the luminosity is changing (e.g. Saslaw, 1978; Mioc and Radu, 1992; Selaru *et al.* 1993), we are in front of a perturbed Maneff-type potential. The two-body problem with equivalent gravitational parameter (see Selaru *et al.* 1992 and the motion in homogeneous potential fields (e.g. McGehee, 1981; Diacu, 1990) belong to the same category. Implications in astrophysics Ureche, 1995), even in atomic physics (see Sommerfeld, 1951; Belenkii 1981; Diacu, 1993), are possible, too.

In this paper we develop the Maneff-type two-body problem, for any value of the field parameters (A, B), and for the whole allowed interplay among these ones, the angular momentum and the total energy. The framework is reduced to a central force problem, for which the analytic solution can be obtained in closed form. The central part of the paper consits of an analysis with qualitative character based on the geometric represantation of the motion in the phase plane. Treating separately the nonradial motion and the radial one, each allowed trajectory in the phase plane is interpreted in terms of physical motion. The possible scenarios are eleven for nonradial motion, and nine for radial motion. Leaving aside the field-free case, these scenarios illustrate three general trends (motion ending in collision, periodic or quasiperiodic orbits, escape trajectories) and two special ones (equilibria and orbits tending to equilibria).

2. ANALYTIC APPROACH

Consider the Maneff-type two-body problem and let M and 1 be the masses. The problem may be reduced to a central force problem, by studying the motion of the unit mass (hereafter particle) in a fixed frame originated in M (hereafter centre). This relative motion will be planar and described by the equation

$$\ddot{r} = -\frac{A}{r^3}\mathbf{r} - \frac{2B}{r^4}\mathbf{r} \ , \qquad (1)$$

where $\mathbf{r}$=relative radius vector of the particle with respect to the centre, $r = |\mathbf{r}|$, and dots signify time-differentiation.

In polar coordinated (r, u), eq. (1) becomes

$$\ddot{r} - r\dot{u}^2 = -\frac{A}{r^2} - \frac{2B}{r^3} \ , \qquad (2)$$

$$r\ddot{u} + 2\dot{r}\dot{u} = 0 \ , \qquad (3)$$

system to which we attach the initial conditions

$$\begin{aligned}&(r, u, \dot{r}, \dot{u})(t_0) = \\ &(r_0, u_0, \dot{r}_0 = V_0\cos\alpha, \dot{u}_0 = V_0\sin\alpha/r) \ ,\end{aligned} \qquad (4)$$

where $V_0 = V(t_0)$, $V = |\dot{\mathbf{r}}|$=velocity, α=angle between initial radius vector and initial velocity.

Two first integrals are easily obtainable. The force being central, the angular momentum is conserved, and (3) provides the first integral

$$r^2\dot{u} = C \ , \qquad (5)$$

where $C = r_0V_0\sin\alpha$ is the constant of the angular momentum. The first integral of energy reads

$$V^2 \equiv \dot{r}^2 + r^2\dot{u}^2 = \frac{2A}{r} + \frac{2B}{r^2} + h \ , \qquad (6)$$

where $h = V_0^2 - 2A/r_0 - 2B/r_0^2$ is the constant of energy.

The analytic solution of the solution can be obtained in closed form. For instance, if the motion is nonradial ($C \neq 0$), resorting to the usual technique, eqs (2) and (5) lead to the Binet-type equation

$$\frac{\mathrm{d}^2(1/r)}{\mathrm{d}u^2} + \left(1 - \frac{2B}{C^2}\right)(1/r) = \frac{A}{C^2} \ , \tag{7}$$

with the initial conditions (Written in an equivalent form extracted from (4) and (5))

$$(1/r,\ \mathrm{d}(1/r)/\mathrm{d}u)(u_0) = (1/r_0,\ -\dot{r}_0/C) \ . \tag{8}$$

The general solution of the initial value problem attached to eq. (7) depends on the sign of the parameter $(1 - 2B/C^2)$, and will be respectively for: (a) $C^2 < 2B$; (b) $C^2 = 2B$; (c) $C^2 > 2B$:

$$r = \left[\left(\frac{1}{r_0} + \frac{A}{2B - c^2}\right)\tilde{C}_u - \frac{\dot{r}_0}{\sqrt{2B - C^2}}\tilde{S}_u - \frac{A}{2B - C^2}\right]^{-1} \tag{9}$$

$$r = \left[\frac{A}{2C^2}(u - u_0)^2 - \frac{\dot{r}_0}{C}(u - u_0) + \frac{1}{r_0}\right]^{-1} \ ; \tag{10}$$

$$r = \left[\left(\frac{1}{r_0} - \frac{A}{C^2 - 2B}\right)C_u - \frac{\dot{r}_0}{\sqrt{C^2 - 2B}}S_u + \frac{A}{C^2 - 2B}\right] \ , \tag{11}$$

with the abbreviations

$$(S_u,\ C_u) = (\sin,\ \cos)\left(\sqrt{1 - 2B/C^2}(u - u_0)\right) \ ,$$

$$(\tilde{S}_u,\ \tilde{C}_u) = (\sinh,\ \cosh)\left(\sqrt{2B/C^2 - 1}(u - u_0)\right) \ .$$

The radial case ($C = 0$) can also be solved, resorting to eq. (6) with $\dot{u} = 0$, but we shall not dwell upon it for two reasons: on the one hand, what we obtain is the dependence $t = t(r)$, relation invertible only in particular cases; on the other hand, we are more interested in an investigation with qualitative character about the particle behavior. Anyway, our qualitative approach (Sections 3-5) will include both cases.

3. QUALITATIVE APPROACH FOR NON-RADIAL MOTION

Eliminating $\dot{u}$ between eqs. (5) and (6), we get

$$\dot{r}^2 = h + \frac{2A}{r} + \frac{2B - C^2}{r^2} \ , \tag{12}$$

which contitutes the basis for our qualitative approach. The cases in which the real motion is *not* possible (those leading to $\dot{r}^2 < 0$) are easily removable; they are

$$\{C^2 = 2B,\ A < 0,\ h \le 0\},$$
$$\{C^2 = 2B,\ A = 0,\ h < 0\},$$
$$\{C^2 > 2B,\ A \le 0,\ h \le 0\},$$
$$\{C^2 > 2B,\ A > 0,\ h < h_{cr} \equiv A^2/(2B - C^2)\}.$$

For real motion ($\dot{r}^2 \ge 0$), eq. 12) allows the construction of the trajectories in the phase plane $(r,\ \dot{r})$.

REMARK 1. By (5), since $C \ne 0$, $\dot{u}$ preserves its sign all along the trajectory. The monotonic variation of u makes an arc of phase curve in the lower/upper phase halfplane $(r, \dot{r}$ represent (physically) spiral motion inwards/outwards. (There is however one exception: if A=0, B=0, namely in the absence of the field, such arcs will physically correspond to rectilinear, *nonradial* motion). Accordingly, a closed phase curve will represent either a periodic (closed) orbit, rosette-shaped (ellipses included), or rather a quasiperiodic (unclosed) orbit filling densely an annulus (see e.g. Arnold, 1976), while a critical phase point will physically mean circular motion.

Constructing the phase curves for the whole allowed interplay among field parameters, angular momentum and total energy, we obtained Figures 1-5, as followe:

	$C^2 < 2B$	$C^2 = 2B$;	$C^2 > 2B$
$A < 0$	*Fig.* 1	*Fig.* 3	*Fig.* 3
$A = 0$	*Fig.* 2	*Fig.* 4	*Fig.* 3
$A > 0$	*Fig.* 2	*Fig.* 2	*Fig.* 5

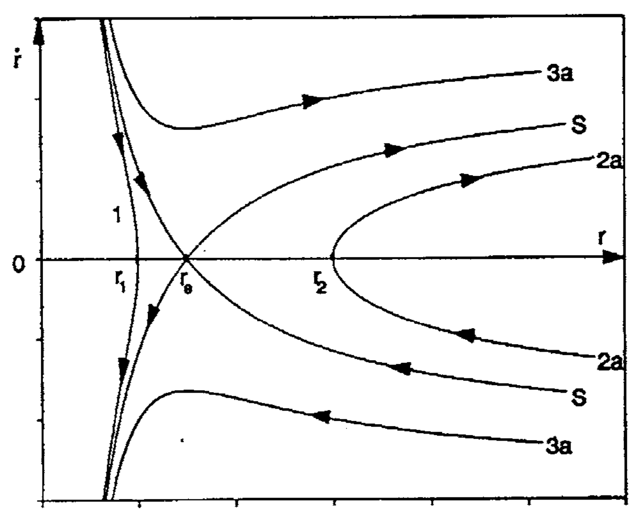

Fig. 1.

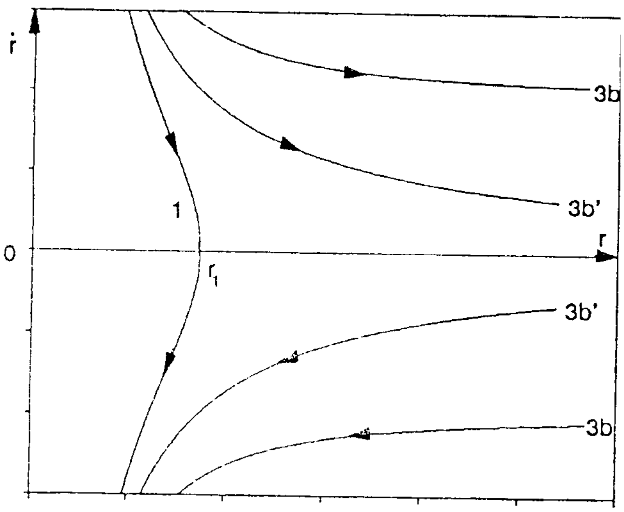

Fig. 2.

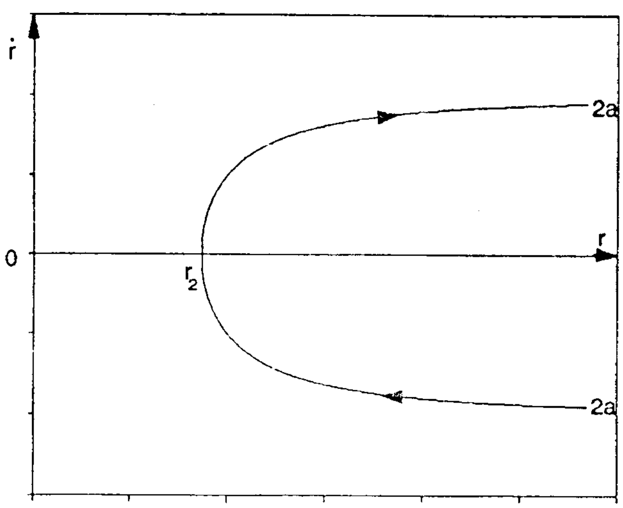

Fig. 3.

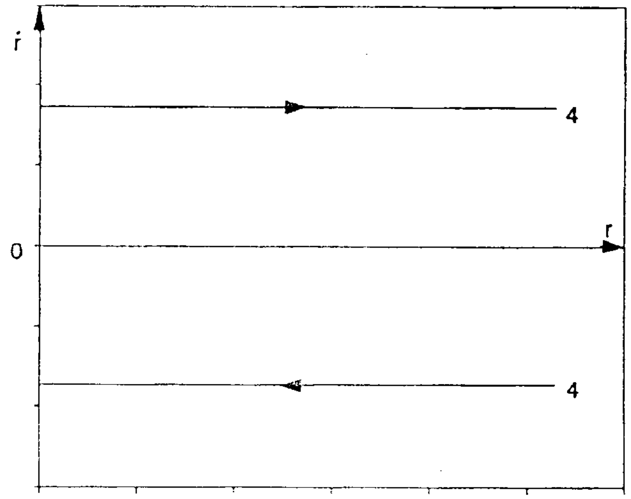

Fig. 4.

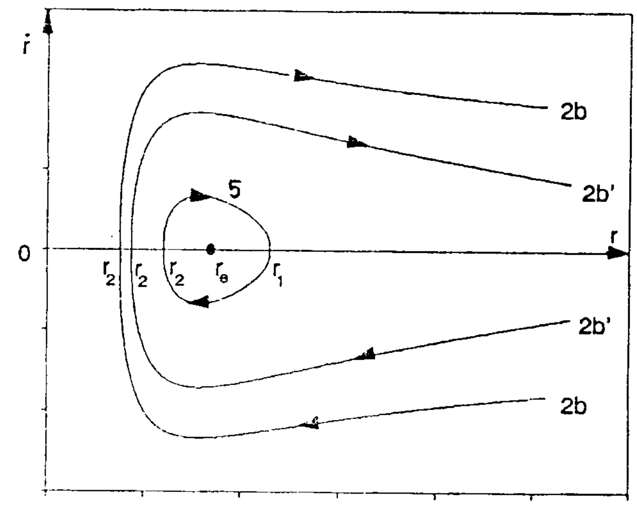

Fig. 5.

Examining these figures, and taking into account the above remark, we may state

PROPOSITION 1. *Consider the two-body problem in Maneff-type field. The only possible scenarios for the relative motion with given nonzero angular momentum are:*

S_1: *spiral motion inwards (possibly coming from infinity) ending in a collision;*

S_2: *spiral motion outwards (possibly beginning by ejection) up to* $r = r_1$, *then spiral motion inwards ending in collision;*

S_3: *spiral motion inwards possibly coming from infinity) tending asymptotically to circular motion of radius* r_e;

S_4: *periodic (rosette) or quasiperiodic motion, starting inwards, confined to the annulus of radii* r_1 *and* r_2;

S_5: *circular motion of radius* r_e;

S_6: *periodic (rosette) or quasiperiodic motion, starting outwards, confined to the annulus of radii* r_1 *and* r_2;

S_7: *spiral motion outwards (possibly beginning by ejection) tending asymptotically to circular motion of radius* r_e;

S_8: *spiral motion inwards (possibly coming from infinity) up to* $r = r_2$, *then spiral motion outwards leading to escape;*

S_9: *spiral motion outwards (possibly beginning by ejection) leading to escape;*

S_{10}: *rectilinear, nonradial motion (possibly coming from infinity) approaching the centre, then tending to infinity;*

S_{11}: rectilinear, nonradial motion with receding from the center and tending to infinity;
where r_1, r_2, r_e are finite and positive quantities determined by field parameters, angular momentum and total energy.

Proof. Introducing the abridging notation

$$\tilde{r}_{1,2} = \frac{-A \mp \sqrt{A^2 - (2B - C^2)h}}{h} \ ,$$
$$\hat{r}_{1,2} \equiv \mp \frac{\sqrt{(C^2 - 2B)h}}{h} \ , \ r_e \equiv \frac{C^2 - 2B}{A} \ ,$$

let us survey the phase curves for every allowed combination $\{A, B, C, h\}$.

If $C^2 < 2B$, $A < 0$, the phase portrait is drawn in Figure 1. We have: for $h \leq 0$ curves 1 with $r_1 = \tilde{r}_1$ ($h < 0$) or $r_1 = r_e/2$ ($h = 0$; for $0 < h < h_c$ curves 1 (if $r_0 < r_e$) with $r_1 = \tilde{r}_1$, or 2a (if $r_0 > r_e$) with $r_1 = \tilde{r}_2$; for $h = h_c$ the separatrix S with the saddle point at $(r_e, 0)$; for $h > h_c$ curves 3a.

If $C^2 < 2B, \ A \geq 0$, the phase curves are those of Figure 2, namely: for $h < 0$ curves 1 with $r_1 = \hat{r}_1$ ($A = 0$) and $r_1 = \tilde{r}_1$ ($A > 0$); for $h = 0$ curves 3b'; for $h > 0$ curves 3b.

If $C^2 = 2B, \ A < 0$ ($h < 0$), the phase curves are 2a in Figure 3 with $r_2 = -2A/h$. If $C^2 = 2B$, $A = 0$ ($h \geq 0$), the phasis curves are given in Figure 4; for $h = 0$ the Orsemiaxis (consisting only of stable equilibrium points); for $h > 0$ the halflines 4. If $C^2 = 2B$, $A > 0$, the phase portrait is drawn in Figure 2 with the same curves for the same energy levels (but with $r_1 = -2A/h$).

If $C^2 > 2B, \ A \leq 0$, ($h > 0$), the phase curves are those given in Figure 3, but with $r_2 = \tilde{r}_2$ ($A < 0$) or $r_2 = \hat{r}_2$ ($A = 0$).

Lastly, if $C^2 > 2B, \ A > 0$ ($h \geq h_c$), we have phase portrait in Figure 5 as follows: for $h_c < h < 0$ curves 5 with $r_1 = \tilde{r}_1$, $r_2 = \tilde{r}_2$; for $h = 0$ curves 2b' with $r_2 = r_e/2$; for $h > 0$ curves 2b with $r_2 = \tilde{r}_2$.

Now, using Figures 1-5, and taking into account Remark 1, let us identify the scenarios implied by every such phase curve. On curves 1 ($r \leq r_1$), $\dot{r}_0 \leq 0 \Rightarrow S_1$, $\dot{r}_0 > 0 \Rightarrow S_2$. On curves 2 ($r \geq r_2$), $\dot{r}_0 < 0 \Rightarrow S_8$, $\dot{r}_0 \geq 0 \Rightarrow S_9$, except the case $A = 0$, $B = 0$ (Figure 3; see also Remark 1), for which $\dot{r} < 0 \Rightarrow S_{10}$, $\dot{r}_0 \geq 0 \Rightarrow S_{11}$. On curves 3 and 4, $\dot{r}_0 < 0 \Rightarrow S_1$, $\dot{r} \geq 0 \Rightarrow S_9$. On curves 5 ($r_2 \leq r \leq r_1$, $\{\dot{r}_0 < 0$ or $r_0 = r_1\} \Rightarrow S_4$, $\{\dot{r} > 0$ or $r_0 = r_2\} \Rightarrow S_6$. The stable point $(r_e, 0)$ in Figure 5 and Or-semiaxis in Figure 4 imply S_5 with stable circular motion of radius $r_e = r_0$. On the separatrix S in Figure 1, with its saddle point at $(r_e, 0)$, $\{r_0 < r_e, \ \dot{r}_0 < 0\} \Rightarrow S_1$; $\{r_0 < r_e, \ \dot{r}_0 > 0\} \Rightarrow S_7$; $\{r_0 > r_e, \ \dot{r}_0 < 0\} \Rightarrow S_3$; $\{r_0 > r_e, \ \dot{r}_0 < 0\} \Rightarrow S_9$; $r_0 = r_e \Rightarrow S_5$ (the corresponding circular orbit being unstable).

Since these phase curves are the only allowed and the corresponding scenarios are the only possible, Proposition 1 is proved.

REMARK 2. Although the motions on the phase curves 2 have the same general features, we have differentiated them according to the asymptotic velocity at infinity: on curves 2a, $V \nearrow \sqrt{h}$, except the case $A = 0$, $B = 0$ (Figure 3, see also Remark 1), for which $V = \sqrt{h}$; on curves 2b, $V \searrow \sqrt{h}$ (=o for 2b'). Exactly the same differentiation was made for the curves 3.

REMARK 3. In general, at ejection/collision $\dot{r} \to \pm\infty$, except for the case $C^2 = 2B$, $A = 0$ (curves 4, Figure 4), for which $r = \pm\sqrt{h}$.

REMARK 4. According to eq. (11), the motion corresponding to S_4 and S_6 is quasiperiodic if $\sqrt{1 - 2B/C^2}$ is irrational, and periodic in the opposite situation (see also Arnold, 1976; Diacu *et al.* 1995).

4. ANOTHER PROOF OF PROPOSITION 1

Let $V_r = \dot{r}$, $V_u = r\dot{u}$ be the polar components of the velocity. It is easy to see that, for $C \neq 0$, eqs. (5) and (6) lead to

$$\frac{C^2 - 2B}{C^2} V_u^2 - \frac{2A}{C} V_u + V_r^2 = h \qquad (13)$$

which represents in the (V_u, V_r)–plane a family of conic sections, whose kind (ellipses, parabolas, hyperbolas) and nature (nondegenerate or degenerate) are given by the parameters $\delta = (C^2 - 2B)/C^2$ and $\Delta = [h(2B - C^2) - A^2]/C^2$, respectively. Observe that we recover the critical value $h_c = A^2/(2B - C^2)$ for which, here $\Delta = 0$ (degenerate conic sections).

If $C^2 < 2B$ we have $\delta < 0$, and eq. (13) represents in this case a family of hyperbolas with center $(-AC/(2B - C^2), 0)$ and foci lying on the V_u–axis if $h < h_c$ ($\Delta < 0$). For $h = h_c$ (13) represents the respective asymptotes, and the family of conjugate hyperbolas for $h > h_c$.

If $C^2 = 2B$ then $\delta = 0$, and eq. (13) represents a family of parabolas, nondegenerate for $A \neq 0$. For $A = 0$ we have $\Delta = 0$ and every parabola reduces to a couple of straight lines (distinct or not) parallel to the V_u–axis.

If $C^2 > 2B$ then $\delta > 0$, and eq. (13) represents a family of ellipses with the same center and semiaxes as the above hyperbolas. The ellipses are real for $h > h_c$ ($\Delta < 0$), reduce to the center of the family for $h = h_c$, and are imaginary for $h < h_c$.

Choosing for C only positive values, the real motion will be possible only in the halfplane V_u ($= C/r) > 0$). With this restriction we recover all combinations $\{A, B, C, h\}$ leading to impossible real motion, interpreted as: $\{C^2 = 2B, \ A < 0, \ h \leq 0\}$–parabolas lying wholly in the forbidden halfplane;

$\{C^2 = 2B,\ A = 0,\ h < 0\}$–imaginary parallel straight lines; $\{C^2 > 2B,\ A \leq 0,\ h \leq 0\}$–real ellipses lying wholly in the forbidden halfplane; $\{C^2 > 2B,\ A > 0,\ h < h_{cr}\}$–imaginary ellipses.

The trajectories in the (V_u, V_r)–plane being only conic sections (degenerate or not) or portions of them in the allowed halfplane, the motion on these curves may have only the following characteristics:

- monotonic increase/decrease of V_u, tending to $\infty/0$;
- monotonic increase/decrease of V_u up to a maximum/minimum value, then monotonic decrease/increase, tending to $0/\infty$;
- monotonic increase/decrease of V_u, tending asymptotically to a finite, positive limit;
- oscillation of V_u between two finite and positive limit values;
- constancy of V_u.

Recall now that $V_u = C/r$, so increase/decrease of V_u means decrease/increase of r; $V_u \to \infty$ means collision (if $V_r < 0$) or ejection (if $V_r > 0$); $V_u \to 0$ means escape. Also observe that, since $\dot{u} > 0$ all along the motion, to every segment of monotonic increase/decrease of V_u in velocity plane corresponds a spiral motion inwards/outwards of the particle (except the case $A = 0$, $B = 0$, mentioned in Remark 1). Accordingly, the oscillation of V_u between two finite, positive, limits means (physically) quasiperiodic or periodic orbits inside an annulus. Lastly, the constancy of V_u means circular motion.

With this interpretation of the allowed scenarios for the motion in velocity plane, scenarios $S_1 - S_{11}$ for real motion pointed out in Proposition 1 are immediately recoverable.

REMARK 5. The use of the velocity plane facilitates the proof of Proposition 1 (the trajectories in this plane being conic section, the picture of the qualitative behavior of the particle follows fairly immediately). However, the radial motion ($C = 0$) cannot be studied in this way.

5. QUALITATIVE APPROACH FOR RADIAL MOTION

Consider now the motion is rectilinear ($C = 0$). In this case, the behavior of the particle is described by

PROPOSITION 2. *Consider the two-body problem in Maneff-type field. The only possible scenarios for the relative motion with given zero angular momentum are:*

S_1*: radial motion inwards (possibly beginning by ejection) ending in collision;*

S*–2: radial motion outwards (possibly beginning by ejection) up to* $r = r_1$*, then radial motion inwards ending in collision;*

S_3*: radial motion inwards (possibly coming from infinity) tending asymptotically to rest at distance* r_e*;*

S_4*: radial libration, starting inwards, between* r_1 *and* r_2*;*

S_5*: rest at distance* r_e*;*

S_6*: radial libration, starting outwaerds, between* r_1 *and* r_2*;*

S_7*: radial motion outwards (possibly beginning by ejection) tending asymptotically to rest at distance* r_e*;*

S_8*: radial motion inwards (possibly coming from infinity) up to* $r = r_2$*, then radial motion outwards leading to escape;*

S_9*: radial motion outwards (possibly beginning by ejection) leading to escape;*

where r_1*,* r_2*,* r_e *are finite and positive quantities determined by field parameters and total energy.*

Proof. Since we did not impose restrictions to C while constructing the phase curves, the proof of Proposition 1 remains valid for $C = 0$, too (observing however that the case $A = 0$, $B = 0$ corresponds now to Figure 4). The phase portraits are the same but, as regards their physical interpretation, Remark 1 transforms into:

REMARK 6. By (5), since $C = 0$, u is constant all along the motion, this makes an arc of phase curve in the lower/upper halfplane $(r, \dot{r})$ represent (physically) radial motion of the particle performed inwards/outwards. Accordingly, a closed phase curve will represent radial libration, while a critical phase point will physically mean rest.

With this, the statement of Proposition 1 (already proved) can be immediately transposed into that of Proposition 2. Scenarios S_{10} and S_{11} are no longer possible, because the motion is now radial.

REMARK 7. All cases concerning comparison between C^2 and $2B$ in the proof of Proposition 1 reduce now to cases concerning the sign of B (it is the same for the forbidden domains. Also, the right-hand side of eq. (12) and subsequently r_1, r_2, r_e change their expressions (by putting $C = 0$).

REMARK 8. Within the framework of the Maneff-type two-body problem, the collision/ejection can be both rectilinear and nonlinear (not only rectilinear, as in the Newtonian problem). In the nonrectilinear case, the particle spirals around the center, performing infinitely many rotations before collision (after ejection); this is the so-called black hole effect (see Diacu *et al.* 1995).

REMARK 9. The qualitative behavior of the particle can also be investigated using the zero relative velocity curves, that are circles whose radii are given by the roots of the equation resulting by putting $V = 0$ in (6). Studying the nature and the sign of these roots, one finds the domains of allowed motion; further, taking into account the characterstics of the motion, the above described scenarios of the motion are recovered.

All these results presented in Sections 2-5 offer a wide picture of the two-body problem in Maneff-type fields from different standpoints: analytic, geometric, and physical.

Acknowledgments – The authors are indepted to Prof. F.-N. Diacu for stimulating discussions and constant support in this research.

REFERENCES

Arnold, V. I.: 1976, *Les methodes matematiques de la mecanique classique*, Mir. Moscou.
Belenkii, I. M.: 1981, *Celest. Mech.* **23**, 9.
Casasayas, J., Fontich, E., Nunes, A.: 1993, in E, Lacomba and J. Libre (eds), *Hamiltonian Systems and Celestial Mechanics*, Advanced series in Nonlinear Dynamics, Vol. 4, World Scientific, Singapore, 35.
Delgado, J., Diacu, F.-N., Lacomba, E. A., Mingarelli, A., Mioc, V., Perez, E., Stoica, C.: 1996, *J. Math. Phys.* **37** (to appear).
Diacu, F.-N.: 1990, *Libertas Math.* **10**, 161.
Diacu, F.-N.: 1993, *J. Math. Phys.* **34**, 5671.
Diacu, F.-N.: 1996, *J. Diff. Eq.* (to appear).
Diacu, F.-N., Mingarelli, A., Mioc, V., Stoica, C.: 1995, in R. P. Agarwal (ed.), *Dynamical Systems and Applications*, World Scientific Series in Applicable Analysis, Vol. 4, World Scientific, Singapore, 213.
Goldstein, H.: 1980, *Classical Mechanics*, 2nd ed., Addison-Wesley, Reading, Massachusetts.
Hagihara, T.: 1975, *Celestial Mechanics*, Vol. 2, Part 1, MIT Press, Cambridge, Massachusetts.
Lacomba, E. A., Llibre, J., Nunes.: 1991, in T. Ratiu (ed.), *The Geometry of Hamiltonian Systems*, Proc. of a workshop held in June 5-16, 1989, Springer-Verlag, Berlin.
McGehee, R.: 1981, *Comment. Math. Helvetici* **56**, 524.
Maneff, G.: 1924, *C. R. Acad. Sci. Paris* **178**, 2159.
Maneff, G.: 1925, *Z. Phys.* **31**, 786.
Maneff, G.: 1930a, *C. R. Acad. Sci. Paris* **190**, 963.
Maneff, G.: 1930b, *C. R. Acad. Sci. Paris* **190**, 1374.
Mioc, V.: 1994, *Astron, Nachr.* **315**, 175.
Mioc, V. Radu, E.: 1992, *Astron. Nachr.* **313**, 353.
Mioc, V., Stoica, C.: 1995a, *C. R. Acad. Sci. Paris* **320**, ser. I, 645.
Mioc, V., Stoica, C.: 1995b, *C. R. Acad. Sci. Paris* **321**, ser. I, 961.
Mioc, V., Stoica, C.: 1995c, *Bull. Astron. Belgrade* **152**, 43.
Saslaw, W. C.: 1978 *Astrophys. J.*, **226**, 240.
Sommerfeld, A.: 1951, *Atombau und Spektrallinien*, Bd. 1, Braunschweig.
Selaru, D., Cucu-Dumitrescu, C., Mioc, V.: 1992, *Astron. Nachr.* **313**, 257.
Selaru, D., Cucu-Dumitrescu, C., Mioc, V.: 1993, *Astrophys. Space Sci.* **202**, 11.
Ureche, V.: 1995, *Rom. Astron. J.* **5** (to appear).

О ПРОБЛЕМУ ДВА ТЕЛА У ПОЉИМА MANEFF-ОВОГ ТИПА

C. Stoica[1] and V. Mioc[2]

[1]*Institute for Gravitation and Space Sciences, Laboratory for Gravitation, Bd. Ana Ipatescu 21, 71111 Bucharest, Romania*

[2]*Astronomical Institute of the Romanian Academy, Astronomical Observatory Cluj-Napoca, Str. Ciresilor 19, 3400 Cluj-Napoca, Romania*

УДК 52-423
Оригинални научни рад

Проблем два тела у пољима Mameff-овог типа (окарактерисаних потенцијалом облика $A/r + B/r^2$; r–растојање између честица, A и B константе) представља добар модел за разне астрономске (и такође физичке, астрофизичке, механичке) проблеме. Релативно кретање у таквим пољима разматра се са аналитичке, геометријске и физичке тачке гледишта. Спроведено је истраживање квалитативног карактера, засновано на испитивању облика фазе, за целу дозвољену област спреге између параметара поља, угаоног момента и укупне енергије. Свака дозвољена трајекторија у фазној равни је интерпретирана помоћу физичког кретања.

Разматрајући одвојено нерадијалне трајекторије, сви могући сценарији, једанаест у нерадијалном случају, а девет у радијалном, су идентификовани. Они илуструју три општа понашања (кретање се завршава сударом, периодичне/квазипериодичне орбите и путање разлаза) и два специјална (равнотежне и орбите које теже ка равнотежи).