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ON THE ORBITS IN NONSTATIC SPHERICALLY SYMMETRIC FIELDS IN ROSEN'S BIMETRIC GRAVITATION THEORY

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SUMMARY: Some general properties of bound orbits and of orbits of capture in spherically symmetric fields in Rosen's theory are studied throught basic inequalities implied. Homogeneous coordinates are used.

1. INTRODUCTION

This paper is a continuation of the previous one (Lukačević, 1994), published in this Bulletin. The essential contibution here is the analysis of some general characteristics of planetary orbits in a spherically symmetric gravitational field in Rosen's theory.

As different from both Newtonian and classically relativistic celestial mechanics, Rosen's theory, as was shown in Lukačević (1986), Lukačević and Ćatović (1992), allows nonstatic spherically symmetric gravitational fields. These fields are induced through a conformal factor, necessarily dependent upon time and st least one space-like coordinate. That coordinate has been chosen to be a modified form of the radial distance from the gravitational source (Lukačević, 1994; Lukačević, 1986), which is almost identical with the geometric radial distance for sufficiently distant bodies. That assumption is physically justified. The conformal factor, which is, in Rosen's theory, a solution of the wave equation, reduces to a mere constant in classical relativity.

In the second section we formulate both Rosen's spherically symmetric line element and the well known Schwarzschild line element. Then the Schwarzschild metric is trensformed to homegeneous coordinates in order to get the relativistic Binet formula in that system. That is done because Rosens metric is formulated in homogeneous coordinates only. In the third section we formulate the geodesic equations for Rosen's nonstatic metric, wherefrom we obtain the corresponding Binet's formula and analyze it for bound orbits and for orbits of capture. The static case in Rosen's theory has not been taken into consideration as it has been studied by Z. Ćatović (1992), thoroughly and with the use of computer simulation.

2. STATIC FIELDS

The metric element in Rosen's bimetric gravitation theory reads

$$ds^2 = e^{2M/r}(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\lambda^2) - e^{-2M/r}dt^2. \qquad (2.1)$$

The space-like part of this metric is, as can be seen, conformally equivalent, through a conformal factor $exp(2Mr^{-1})$, to the Euclidian metric element, expressed with respect to a spherical coordinate system. This means that the coordinates used are isotropic. In order to dtudy analogies between two theories, let us write first the Schwarzschild line element in its classical form

$$ds^2 = \frac{dR^2}{1-\frac{2m}{R}} + R^2(d\vartheta^2 + \sin^2\vartheta d\lambda^2) - (1-\frac{2m}{R})dt^2. \tag{2.2}$$

R is the radial distance form the central gravitational source to a point in space, $2m$ is the gravitational radius. If one performs the transformation $R \rightarrow r$

$$R = r(1+\frac{m}{2r})^2 \tag{2.3}$$

the result will be a spherically symmetric line element of the form

$$ds^2 = (1+\frac{m}{2r})^4(dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\lambda^2) - \left(\frac{1-m/2r}{1+m/2r}\right)^2 dt^2 \tag{2.4}$$

which is analogous, in the classical general relativity, to the metric element (2.1) in Rosen's theory, i.e., it is an isotropic metric element, the variable r being, by (1.3), a function of the radial distance R. It is, in fact, a slightly modified radial distance, as r differs from R by a small quantity; m (and consequently M in Rosen's theory) being small compared to r in the problems of celestial mechanics.

Let us begin with Binet's formula of celestial mechanics in classical relativity (Papapetrou, 1974)

$$\frac{d^2}{d\lambda^2}\left(\frac{1}{R}\right) + \frac{1}{R} = \frac{m}{a^2} + \frac{3m}{R^2} \tag{2.5}$$

where a is the well known constant of the Keplerian integral of motion. If we perform the transformation (1.3) and put, for convinience, $u \equiv r^{-1}$, equation (2.5) will, after some manipulations, take the form

$$u'' - 2m\frac{1-\frac{1}{4}mu}{1-\frac{1}{4}m^2u^2}u'^2 + u\frac{1+\frac{1}{2}mu}{1-\frac{1}{2}mu} = \frac{m(1+\frac{1}{2}mu)^3}{a^2(1-\frac{1}{2}mu)} + \frac{3mu^2}{1-\frac{1}{4}m^2u^2}; (u' \equiv \frac{du}{d\lambda}) \tag{2.6}$$

We note, in the above formula, that it is not only more cumbersome than (2.5), but that a term with u' appears, which was not the case when Binet's formula was expressed with respect to R^{-1}.

In order to analyze (2.6) at the characteristic points of the orbit, we shall establish first the relation between the derivatives (with respect to λ) of R and r. From (2.3) follows immediately

$$R' = r'(1-\frac{m^2}{4r^2}), \quad R'' = r''(1-\frac{m^2}{4r^2}) + \frac{m^2r'^2}{2r^3} \tag{2.7a}$$

Further, by the definition of u

$$u' = -\frac{r'}{r^2}, u'' = \frac{2r'^2 - rr''}{r^3} \tag{2.7b}$$

We see that at the periastra and apoastra, characterized by $R' = 0$ we have $r' = 0$ and R'' is proportional to r'', but we have also $u' = 0$ and u'' is proportional (with changed sign) to r''.

Let us consider now the celestial body at the periastron, By (1.7a-b) its orbit is characterized at this point by

$$u' = 0, u'' < 0$$

wherefrom, by (2.6)

$$\frac{m(1+\frac{1}{2}mu)^3}{a^2(1-\frac{1}{2}mu)} + \frac{3mu^2}{1-\frac{1}{4}m^2u^2} > u\frac{1+\frac{1}{2}mu}{1-\frac{1}{2}mu} \tag{2.8a}$$

and consequently, at the apoastron

$$u' = 0, u'' > 0$$

with

$$\frac{m(1+\frac{1}{2}mu)^3}{a^2(1-\frac{1}{2}mu)} + \frac{3mu^2}{1-\frac{1}{4}m^2u^2} < u\frac{1+\frac{1}{2}mu}{1-\frac{1}{2}mu} \tag{2.8b}$$

The value u_p, corresponding to the periastron, being greater than the value u_a, corresponding to the apoastron, we have therefore

$$u_p > u_a \tag{2.9}$$

3. NONSTATIC FIELDS

In the preceding section we formulated two spherically symmetric fields, (2.1) and (2.2) (or (2.4)), in two different gravitational theories (both static). But we formulated some conditions for the motion of a celestial body in one of them, in classical relativity only.

We shall consider now a nonstatic field in Rosen's theory. By Birkhoff's theorem such a field is not possible in classical relativity. The metric element of a nonstatic spherically symmetric field in Rosen's theory is conformally equivalent to the static metric element (2.1), through a conformal, factor which depends on r and the coordinate time t

$$d\tilde{s}^2 = e^{2\varphi(r,t)}[e^{2M/r}(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\lambda^2) - e^{-2M/r}dt^2] \tag{3.1}$$

where φ is a solution of the wave equation (Lukačević, 1986; Lukačević and Ćatović, 1992.

The differential equations of motion of a celestial body in the equatorial plane of the spherically symmetric field (3.1) are the geodesic equations of that field. They are three altogether, the fourth one, corresponding to ϑ, reduces identically to zero as the orbit lies in the equatorial plane.

The equation which corresponds to the variable λ yields the Keplerian integral of motion.

$$r^2 e^{2[Mr^{-1}+\varphi(r,t)]}\frac{d\lambda}{ds} = l = const. \tag{3.2}$$

Substituting $d\tilde{s}$ by $d\lambda$ in the remaining two differential equations of geodesic one obtains

$$r'' - \frac{2}{r}(1 - \frac{M}{r})r'^2 + 2M - r + l^{-2}r^4 e^{2(Mr^{-1}+\varphi)}\left(\frac{\partial\varphi}{\partial r} + \frac{M}{r^2}\right) = 0 \tag{3.3}$$

$$t'' - \frac{2}{r}(1 - \frac{2M}{r})r't' - l^{-2}r^4 e^{2(3Mr^{-1}+\varphi)}\frac{\partial\varphi}{\partial t} = 0 \tag{3.4}$$

where $f' \equiv \frac{df}{d\lambda}$. The system (3.2)-(3.4), (Lukačević, 1994; Lukačević and Ćatović, 1992) is basic for the study of motion in the relativistic one-body problem. Making use of the variable $u = r^{-1}$ equation (3.3) takes the form

$$u'' - 2(M - \frac{2}{u})u'^2 + u(1 - 2Mu) + l^2 e^{(2Mu+\varphi)}\left(\frac{\partial\varphi}{\partial u} - M\right) = 0. \tag{3.5}$$

It is interesting that (3.5), which corresponds to a time-dependent field, is less cumbersome than the differential equation (2.6) for classically relativistic (i.e. static) gravitational field.

a) Let us consider the extremal points of the orbit. Assuming that such points exist, we shall have, by analogy with (2.8a), (2.8b) and by (2.7a), (2.7b), the following inequalities:

At the periastron

$$u' = 0, u'' < 0 \Leftrightarrow$$
$$\Leftrightarrow u + l^2 e^{2(Mu+\varphi)}\frac{\partial\varphi}{\partial u} > M(2u + l^2 e^{2(Mu+\varphi)}) \tag{3.6a}$$

and consequently, at the apoastron

$$u' = 0, u'' > 0 \Leftrightarrow$$
$$\Leftrightarrow u + l^2 e^{2(Mu+\varphi)}\frac{\partial\varphi}{\partial u} > M(2u + l^2 e^{2(Mu+\varphi)}) \tag{3.6b}$$

The general conditions satisfied by φ, obtained in Lukačević and Ćatović (1992) read

$$\frac{\partial\varphi}{\partial t} < 0; \frac{\partial\varphi}{\partial r} > 0 \tag{3.7}$$

u' being null, by (2.7b) and the second inequality (3.7), one obtains from (3.4)

$$t'' = l^{-2}r^4 e^{2(3Mr^{-1}+\varphi)}\frac{\partial\varphi}{\partial t} < 0 \tag{3.8}$$

That condition ought to be satisfied in both extremal points. This requirement seems somewhat surprising, but it is a consequence of the nonstatic character of the gravitational field. We point out that the left-hand-side of the Keplerian integral (3.2) depends explictly on time, which is also a consequence of the nonstatic metric.

Let us consider formulas (3.6a-b). By the second inequality (3.7) we have

$$\frac{\partial\varphi}{\partial u} = r^{-2}\frac{\partial\varphi}{\partial r} < 0 \tag{3.9}$$

There results, by (3.9), that the magnitude of the second term on the left- hand-side of (3.6a-b) is the determining factor. That term, being negative, reverses the inequality for a sufficiently large r.

b) Let us consider the possibility of motion of the body on an orbit of capture. Such orbits, togather with bounded ones previously considered, have been studied by Ćatović (1992), in the static case. Comparisons have been made in Ćatović (1992), with orbits in Schwarzschild's metric, based on computer simulations. We shall analyze here the qualitative side of the question, the function φ being, by its definition, largely undetermined.

The essential assumption for an orbit of capture is

$$r' < 0 \tag{3.10}$$

during motion. By (3.4) and the first inequality (3.7), as $t' > 0$ necessarily, we obtain

$$t'' > 0 \tag{3.11}$$

which means, as t' is the reciprocal value of φ', that the orbital motion around the central body is accelerated. That is a consequence of the decreasing distance between the moving body and the gravitational source.

The differential equation of radial motion (3.3) is somewhat more complicated to interprete than (3.4). For

$$r - 2M + \frac{2}{r}(1 - \frac{M}{r})r'^2 > l^{-2}r^4 e^{2(Mr^{-1}+\varphi)}\left(\frac{\partial\varphi}{\partial r} + \frac{M}{r^2}\right) \tag{3.12}$$

one has, by (3.3) and (3.7)

$$r'' > 0 \tag{3.13}$$

which means that the motion is, by (3.10), radially decelerated, i.e. the radial fall is slowed down. The inversion of the inequality (3.12) implies the inversion of the inequality (3.13). The inequality (3.12), and consequently (3.13), is obviously easier satisfied for $\varphi < 0$ than the converse case, although this is not in itself a condition in the strict sense.

Let us assume, for simplicity, φ negative and of the form given by the equation (2.11) in (Lukačević, 1994), that is

$$\varphi = -\frac{1}{2}\psi(t - r) \tag{3.14}$$

with further assumptions

$$\psi(t - r) > 0, \frac{\partial\psi(t - r)}{\partial(t - r)} > 0 \tag{3.15}$$

ψ being a positive definite function in space time. φ, defined as above, satisfies the wave equation (Tikhonov and Samarsky, 1953) in the case of waves generated by the central source (outgoing waves), which is the central body. Conditions (3.15) ensure that (3.7) is satisfied. The radial velocity r' of the body moving on the orbit of capture is not assumed very large a priori at least during the period of motion considered. This means that in (3.12), under physically reasonable assumptions, the term $2r^{-1}(1-Mr^{-1})r'^2$ is certainly smaller by its order of magnitude than the term $r - 2m$. Simultaneously, the term r^4 decreases faster than r, whereas the exponential function at the right-hand-side is very nearly unity; the same is true of the term between parentheses. So the inequality becomes stronger during the fall, despite the fact that r' possibly decreases. Finally, we assume that

$$\frac{\partial\psi(t - r)}{\partial(t - r)} << 1 \tag{3.16}$$

i.e. ψ is slowly varying function of its argument, so that the partial derivative of φ at the right-hand-side of (3.12) is small enough to ensure that inequality.

4. CONCLUSION

We restricted ourselves, in this paper, to the qualitative aspect of the motion of a celestial body in a nonstatic spherically symmetric grvitational field. We underline that, in Section 2, only the central field in classical relativity was considered, which is necessarily static, in order to make a comparison between the forms the differential equations of motion take in polar and in homogeneous coordinates respectively. The rather cumbersome formula (2.6), compared to the well-known (2.5), gives an idea of the complications aroused by the transformation (2.3) of the radial coordinate.

One of the interesting features of a nonstatic field is the fact that, in case of periodic orbits, the extremal points, considered in a), i.e. the periastron and the apoastron, satisfy the same inequality (3.8). That is , to our mind, a strong argument in favour of the probabilty of motion of celestial body on orbits of capture, exposed in b), hold at least as long as the respective values of R and r in (2.5) do not differ noticeably, that is, down to a minimal value of r, which is naturally limited by the radius of the central body. This implies, of course, that the minimal value of the radius is noticeably greater than the value which would correspond, in classical relativity, to the "gravitational radius", $r_{min} >> 2M$, but which does not represent a horizon in Rosen's theory.

As we have seen, the inequalities which have to be satisfied for two types of motion, periodic (or bounded) orbits and orbits of capture, are allowed in a nonstatic field. But it is possible that, because of the time dependence of the metric, periodic orbits become unstable after some time, and tend to become orbits of capture. We particularly point out the fact that we chose a negative function φ, as it was assumed to satisfy the condition $r \to \infty, \varphi \to 0$ (which holds for a wave functions (Fock, 1959) and results from (2.14), ψ being a finite function in the whole of space-time). There results that, with such a sign, the second inequality (3.7) is satisfied in the whole space-time, whereas φ, for r and t sufficiently large, becomes necessarily a slowly varying function of its argument.

In general, I conclude that the motion on orbits of capture has to be the object of further detailed study.

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О ОРБИТАМА У НЕСТАТИЧКИМ СФЕРНО СИМЕТРИЧНИМ ПОЉИМА У РОЗЕНОВОЈ БИМЕТРИЧКОЈ ТЕОРИЈИ ГРАВИТАЦИЈЕ

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Оригинални научни рад

Изучавају се неке опште особине затворених орбита и орбита захвата у сферно симетричним пољима у Розеновој биметричкој теорији гравитације. Услови за затворене (периодичне орбите формулишу се у карактеристичним тачкама (периастру и апоастру), док се за орбите захвата поставља општи услов брзини радијалног пада. Указује се на то да су затворене орбите, с обзиром на нестационарност, вероватно нестабилне.