Bull. Astron. Belgrade № 152 (1995), 35 – 42

UDC 521.176
Original scientific paper

PERTURBATION THEORY FOR LOW SATELLITES: AN APPLICATION

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(Received: October 8, 1995)

SUMMARY: A new, fully analytical perturbation theory for low satellites, intended for a preliminary study of their orbits, is presented. This theory uses a mixed Lagrange-Hamilton formalism, separately accounting for the very short-periodic, medium-periodic and long-periodic effects. The truncation, with respect to eccentricity, to degree 1 is justified by the fact that e must be small to avoid hard landing, while the inclination functions have to be handled in full generality. The two main problems are the slow convergence with respect to the degree and order of the spherical harmonics expansion of the gravity field of the central body, and the presence of second order effects. An application of the theory to the mission analysis of a low polar lunar orbiter, such as the proposed European mission MORO, is described.

1. INTRODUCTION

The dynamics of low-altitude satellites present a specific problem for several reasons. One of the most important difficulties is due to the need to accurately predict the time variations of the eccentricity, which must remain within certain limits to prevent the crash of the satellite against the surface of the central body. A powerful tool to study satellite orbits and monitor the variations of the orbital elements is undoubtedly the numerical integration; it gives precise and reliable data on the characteristics and behaviour of an orbit, but is rather inefficient, in particular for some specific purposes. For example, due to the fact that the long-periodic variations of the eccentricity are normally much larger than variations of short periods, to compute long time series of minimum altitude for a multitude of initial conditions and determine the safety zones, is very inefficient. To examine the entire phase space of orbital elements by numerical integration, in order to locate the "frozen orbit", is almost impossible. Hence the need for analytical solutions.

An orbit around some primary body is subject to perturbation of its osculating two-body orbital elements, due to the harmonics of the gravity field of the primary, to the differential attraction from other massive bodies, to non gravitational perturbations, etc. The effects belong to three main classes: very short-periodic ones (with periods of the order of the satellite orbital period, i.e. a few hours, or less), medium-periodic (with periods longer than one orbital period of the satellite, but shorter than one period of rotation of the primary), and long-periodic; very short and medium periodic perturbations will be collectively called short periodic. The theory takes a much simpler form if it is possible to neglect the resonances between short and medium-periodic terms, as it is the case for a slowly rotating central body, such as the Moon.

There are three types of elements involved: osculating, mean and proper. Osculating elements are the instantaneous ones; they can be expressed as a set of orbital elements (e.g. keplerian), by a time independent coordinate transformation from the state vector (cartesian position and velocity). Mean elements are obtained from the osculating ones by removing all the perturbations with short periods. Proper elements are obtained by removing from the mean elements the long-periodic perturbations.

Proper elements are solution of an integrable problem, whose time evolution can be computed analytically and with a simple formula. However, the transformation of a non integrable problem into an integrable one cannot be performed in an exact way, but only by neglecting some higher degree and order terms. Thus, the proper elements, which should be constant in the integrable dynamics, are not exactly constant when computed from a time series of state vectors. Following a procedure well established for asteroid proper elements, we use the standard deviation of these proper elements with respect to their long term average as a measure of the accuracy of the proper elements theory (see Milani and Knežević, 1990, 1992, 1994).

We are developing fully analytical perturbation theories to compute proper elements for low-altitude satellites (Milani and Knežević 1995, Knežević and Milani 1995), which can serve as an efficient and handy tool for the preliminary study of their orbits. We are using a mixed Lagrange-Hamilton formalism, separately accounting for the very short-periodic perturbations, the medium-periodic perturbations, and the long-periodic ones. For a low satellite, truncation with respect to the degree in eccentricity is legitimate, while the inclination functions have to be handled in full generality. The two main problems are the slow convergence with respect to the degree and order of the spherical harmonics expansion of the gravity field of the central body, and the presence of second order effects (containing the square of the gravitational potential coefficients).

When the theory is applicable, the results are accurate enough to analytically predict the orbit; the computation being fully analytic and explicit, it is much more efficient than any conceivable numerical technique. The proper elements can also be used to study problems such as orbit maintenance and optimum manoeuver cycles, with significant practical implications.

2. THE PERTURBING FUNCTION AND THE CHOICE OF THE TRUNCATIONS

The potential of the gravity field of a non-spherical body (primary) is given in terms of the development into spherical harmonics (Kaula 1966):

$$U = \frac{GM}{r} + R$$

$$R = \frac{GM}{r}\sum_{l=2}^{+\infty}\left(\frac{R_M}{r}\right)^l \sum_{m=0}^{l} P_{lm}(\phi)\,[C_{lm}\cos m\lambda + S_{lm}\sin m\lambda]$$

Here M and R_M are mass and equatorial radius of the primary, r is the distance from its centre, G is the gravitational constant, λ and ϕ are longitude and latitude with respect to some reference system body-fixed with the primary, P_{lm} is the Legendre associated polynomial, and C_{lm}, S_{lm} are cosine and sine coefficients of spherical harmonic potential terms, respectively.

Since the $l = 1$ terms are removed by translation of the origin of the reference system to the centre of mass of the primary, the *perturbing function* R contains only the terms of degree $l \geq 2$. The perturbing function can be expressed as a function of the usual keplerian orbital elements $(a, e, I, \Omega, \omega, \ell)$ (semimajor axis, eccentricity, inclination to the lunar equator, longitude of node, argument of periselenium, mean anomaly), and expanded as follows:

$$R = \frac{GM}{a}\sum_{l=2}^{+\infty}\left(\frac{R_M}{a}\right)^l \sum_{m=0}^{l}\sum_{p=0}^{l} F_{lmp}(I) \sum_{q=-\infty}^{+\infty} G_{lpq}(e) S_{lmpq}(\omega, \ell, \Omega, \theta)$$

$$S_{lmpq} = \begin{cases} C_{lm}\cos\Psi_{lmpq} + S_{lm}\sin\Psi_{lmpq} & (l\text{-}m \text{ even}) \\ -S_{lm}\cos\Psi_{lmpq} + C_{lm}\sin\Psi_{lmpq} & (l\text{-}m \text{ odd}) \end{cases}$$

$$\Psi_{lmpq} = (l-2p)\,\omega + (l-2p+q)\ell + m(\Omega - \theta)$$

where θ is the phase of the rotation of the primary, namely the angle between some body fixed direction along the equator and some inertial direction along the equator. The *inclination functions* F_{lmp} and the *eccentricity functions* G_{lpq} can be explicitly computed.

We can now define very short-periodic terms in R as those with $l-2p+q \neq 0$ (i.e., those containing the mean anomaly ℓ); medium period terms are those with $l-2p+q=0$ but $m \neq 0$ (i.e. those containing $m\theta$); long-period terms have both $l-2p+q=0$ and $m=0$, so that:

$$R = \overline{R} + \hat{R} + \tilde{R}$$

where $\overline{R}$ contains only the long-period terms, $\hat{R}$ only the medium-period terms, $\tilde{R}$ only the very short-period ones. The mean elements are, consequently, such that their equations of motion contain only the derivatives of $\overline{R}$.

The algorithm contains two stages: first the short-periodic perturbations (containing the derivatives of $\hat{R}+\tilde{R}$) are removed, second we truncate $\overline{R}$ to obtain an integrable system, which we can solve in closed form. Both stages will involve neglecting some "terms"; the approximation will be consistent if the neglected terms have smaller effects than those included in the theory; the accuracy is controlled by how small they are. This choice must be done in a different way for each particular case. As an example, we are discussing here a consistent theory for a low lunar polar orbiter.

The first approximation is the truncation of the perturbing potential R as a function of the orbital elements. For a lunar orbiter, the eccentricity e can not be large (to avoid hard landing), while the inclination I can be large (and indeed we are interested in polar orbits, $I \approx 90°$). Hence, we truncate all the perturbations to degree 1 in e; this requires the expansion of the perturbing function to degree 2 in e, since some perturbations contain derivatives such as $\partial R/\partial e$. We also perform some truncation which takes into account that the orbit is nearly polar, that is $\cos I$ is small.

A second truncation is with respect to the degree l in the spherical harmonics expansion. This is justified by the fact that the harmonic coefficients C_{lm}, S_{lm} are decreasing with l, roughly proportionally to $1/l^2$, according to the well known Kaula's rule. Our theory has no *a priori* upper limit l, but some limitation has to be chosen to control the computational cost, and also to avoid numerical instabilities. Moreover the actual values of the high degree and order harmonic coefficients are highly uncertain, and there is no point in doing very long computations based on unreliable input data.

The third truncation is intrinsic to any perturbation theory, and is a truncation to some order in the small parameters (harmonic coefficients) appearing in the perturbations. For the short-period perturbations, a first order theory is accurate enough. For the long-period perturbations, if the accuracy required is very high and the time span is very long, some terms belonging to the second order in the small parameters should be added. The current version of our theory does not include these second order terms, also because the uncertainty of the harmonic coefficients of gravity field of the Moon results in a larger error in the solution.

After all these truncations, the short-period perturbations can be described by a trigonometric series which has many terms but is easy to handle with a computer program. The long-period perturbations are described, in this approximation, by a system of linear differential equations with constant coefficients, with an elementary solution. The equilibrium point of the long-periodic equations corresponds to the so called "frozen orbit", which has no long-period perturbations in eccentricity and inclination: the ideal orbit for a long-lived spacecraft not needing any orbit control.

3. LIMITATIONS OF THE CURRENT THEORY

This theory uses some assumptions and performs some truncations and simplifications with respect to a complete problem. The choices we have made correspond to the requirements of the preliminary mission analysis of a low lunar polar orbiter. However, these assumptions and simplifications ought to be explicitly stated, in order to be able to remove them if the need arises for a theory with higher accuracy and/or more general applicability.

1. Second order effects. We have used a semianalytical theory (that is, numerical integration of the equations of motion resulting from the analytical expansions) to discriminate the impact of the omitted second order effects. The effect can grow up to 0.001 in eccentricity after $\approx 1\,yr$. This source of error was considered unimportant at this stage of development of mission analysis tools, because the uncertainty in the lunar potential results in a much larger uncertainty in the long term behaviour. However, the inclusion of the main second order long-periodic effects is certainly a worthwhile upgrade of our theory, which would become necessary when a better model of the lunar gravity field will be available.
2. Effects not included in the current version. Perturbations by the Earth and the Sun are not accounted for; a theory could be developed for these, as soon as the need arises. Nor does the current theory take into account the effects of radiation pressure, which can be relevant when the lunar satellite undergoes eclipses (Milani et al. 1987): the change in proper semimajor axis can accumulate up to $\approx 50\,m$.
3. The current version of our theory is not suitable for the computation of the perturbations due to very high harmonics, e.g. $l > 40$. To compute the perturbations up to such a high l would not be very useful now, given the present state of the art of the lunar potential models, in which the harmonics of such a high degree mostly reflect the

a priori constraints used in the collocation process. However, when a reliable potential model will be available, it will become necessary to ensure sufficient performance and numerical stability even for high l. Numerical instability problems arise because the coefficients of the monomials in $\sin I$ in F_{lmp} grow very fast with l; e.g. these coefficients become larger than the inverse of the machine error for $l \simeq 50$. The resulting numerical instability could be avoided by expanding F_{lmp} in a neighbourhood of $I = 90°$.

4. The precession of the lunar pole results in a drift of the inclination in the true of date system (that is, with respect to the current lunar pole). After a few years, the inclination appearing in the coefficients $F_{lmp}(I)$ becomes noticeably different from the one in the true of epoch system we are using, and this results in a degradation of the solutions, because of a less accurate removal of medium periodic perturbations (mostly $m = 1$). This could be improved, if the need arises to cover a longer time span.

4. ACCURACY OF THE RESULTS

We performed two kinds of tests to check whether the results which we achieved are good enough for the purpose of the preliminary study of the satellite orbit (mission analysis). First, we computed the proper elements for a time series of state vectors of a lunar orbiter, and, using the well-known testing scheme developed for asteroids (Milani and Knežević 1990), we estimated the root mean square deviation of the proper values from their averages (proper elements being by definition constants in time, these deviations represent an indirect measure of the accuracy of the results and check of the validity of the theory itself). For integrations made by means of the software system USOC (G. Lecohier, private communication), with the gravity field model by Lemoine et al. (1994), over a time span of 275 days and accounting only for the perturbations due to harmonics of the gravity field of the Moon, we found the r.m.s. in semimajor axis, eccentricity and inclination to be $\approx 13\,m$, < 0.001 and $< 0.001\,rad$, respectively, which is more than sufficient for the stated purpose. Similar results were obtained with integrations made by means of the GEODYNE software (R. Floberghagen, private communication) and with gravity field by Konopliv et al. (1993).

The second test was even more demanding: we have computed analytically a solution for the same time span of the numerical test. What we did in fact (see Figure 1.) was to compute the proper elements for the initial instant of the numerical integration, propagate them analytically for the same span of time covered numerically, and recompute the osculating values for these instants of time for which the values are sampled in the numerical integration. Then we computed the difference between the analytical and the numerical solutions. In this case we found that for about 6 months (more than enough with respect to typical duration of a lunar mission) the theory provides solution at the 0.001 level of accuracy in eccentricity, which is entirely satisfactory (see Figure 2.). We can conclude that, although the analytical theory is not meant to provide precise ephemerides of the satellite, but only to study the qualitative long term behaviour of the orbit (e.g. for manoeuver planning purposes), this test shows its capability to actually predict the orbit in a qualitatively correct way and even quantitatively with a reasonable accuracy, again more than it is presumably needed for the preliminary mission analysis. Although this is not of an utmost importance in this context, let us also mention that the analytical propagation by means of proper elements is at least an order of magnitude faster than the numerical integration.

5. MISSION ANALYSIS: AN EXAMPLE

The described theory was originally developed for a practical application, namely the study of the proposed European lunar mission MORO (with launch opportunity in 2002–2003). There are three problems to be solved to design and perform such a mission:

1. The fuel consumption has to be kept as low as possible to reduce the total spacecraft mass. This requires the study of an efficient manoeuver cycle, taking into account the medium and especially long period perturbations.
2. The eccentricity of the orbit has to be kept very low to maintain adequate coverage by the camera: if the mean altitude is 100 Km, an e of 0.02 implies a change of image scale by more than a factor 2 (and an e only slightly larger than 0.05 results in hard landing).
3. MORO is supposed to release a sub-satellite to perform Satellite To Satellite tracking (also to measure the gravity field on the far side of the Moon). The subsat has no orbit control system, and we need to know if and when it would crash on the surface.

Although numerical tests can give some answer to these problems, a systematic exploration of the phase space can be done efficiently only with a fully analytical theory. In Figure 3. we present a possible solution for the choice of orbits and their maintenance, with a manoeuver cycle of about 60 days for MORO and the subsat placed in the the equilibrium point of the dynamics of the mean elements, the frozen orbit. The solution has been computed

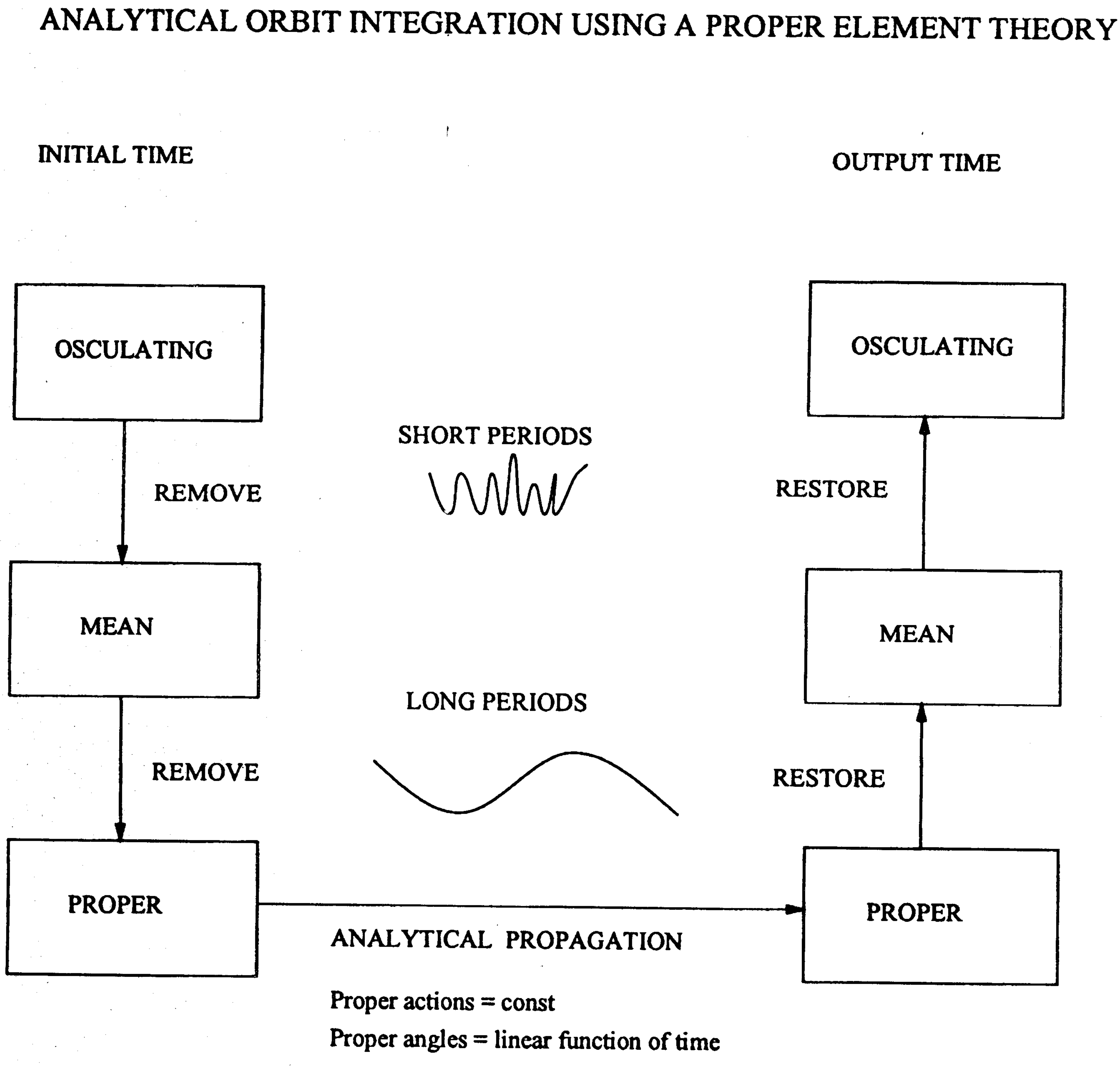

Fig. 1. A schematic representation of the analytical orbit integration by means of a proper element theory.

using the Konopliv et al. (1993) lunar gravity field, truncated to degree and order 20. The manoeuver strategy had the purpose of keeping the main spacecraft within $e < 0.015$, with as little consumption of fuel as possible and the duration of the mission as long as possible, while the sub-satellite's mean elements were to change very little (in the neighbourhood of the frozen orbit). The outer circle represents the "safe periselenium altitude" line, which corresponds to a minimum periselenium altitude of $20\,Km$, taking into account the lunar topography (up to $\simeq 10\,Km$), as well as the inaccuracy of the theory (about $2\,Km$). As one can easily infer from the Figure, conditions 2. and 3. are amply satisfied by the particular orbits, and for a span of time more than enough for the assumed duration of the propo-

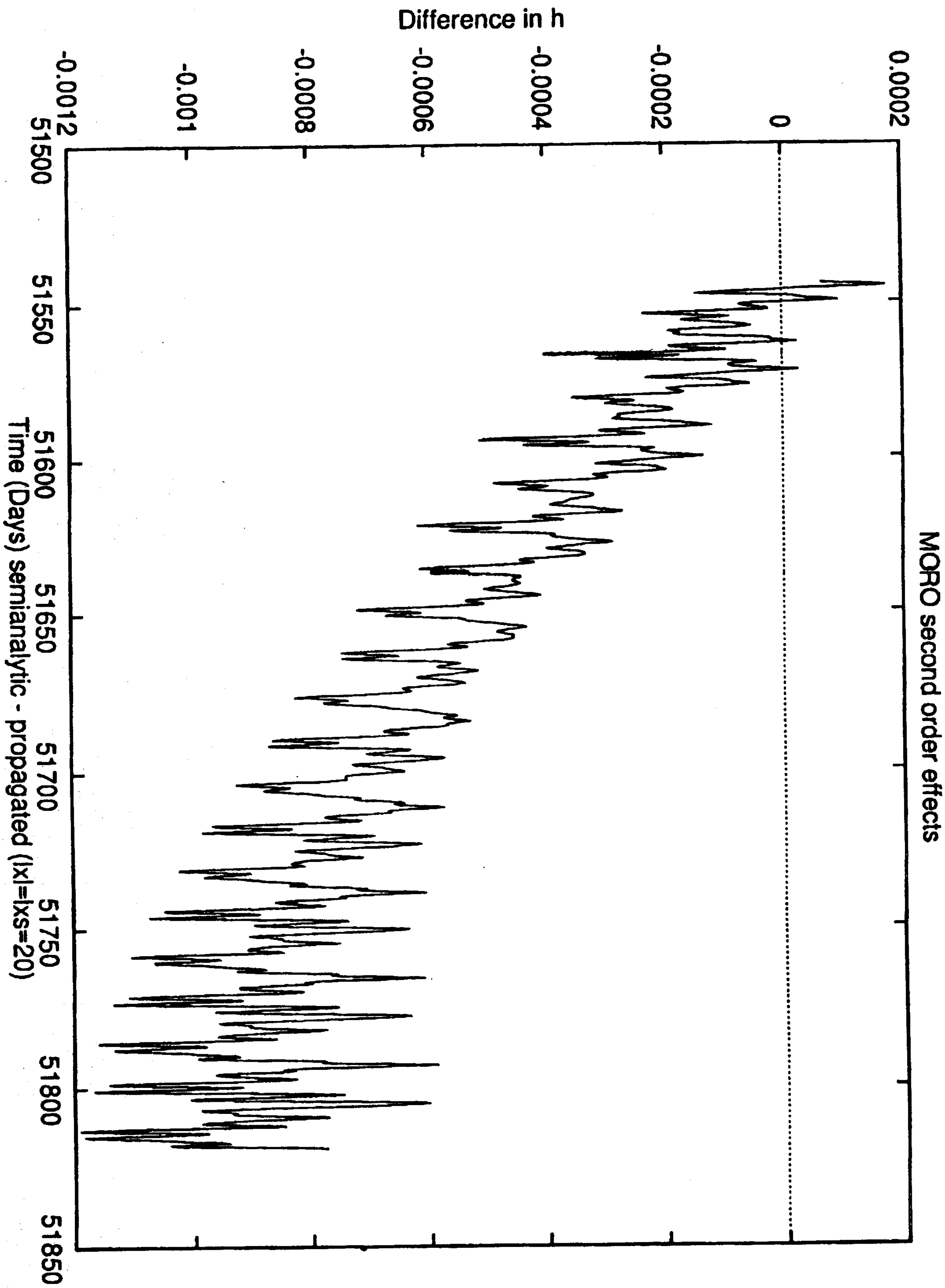

Fig. 2. The difference –in one of the eccentricity related variables, namely $h = e \sin \omega$– between the numerical integration and the purely analytical propagation. Apart from the very short periodic perturbation, the main trend is due both to truncation in the spherical harmonics expansion and to second order effects.

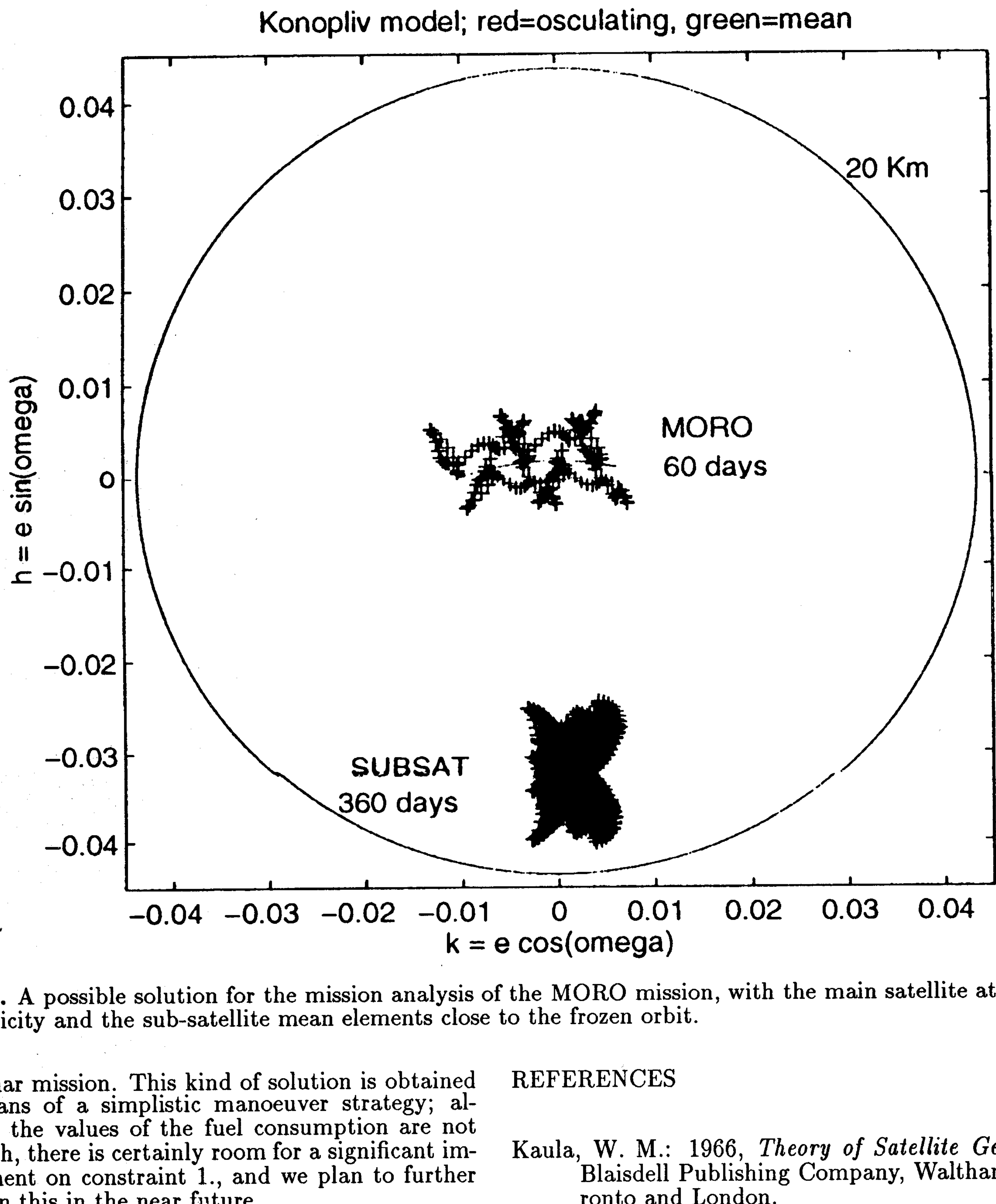

Fig. 3. A possible solution for the mission analysis of the MORO mission, with the main satellite at a low eccentricity and the sub-satellite mean elements close to the frozen orbit.

sed lunar mission. This kind of solution is obtained by means of a simplistic manoeuver strategy; although the values of the fuel consumption are not too high, there is certainly room for a significant improvement on constraint 1., and we plan to further work on this in the near future.

Acknowledgements – This work is a part of the project "Physics and dynamics of celestial bodies", supported by Ministry of Science and Technology of Serbia.

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ЈЕДНА ПРИМЕНА ТЕОРИЈЕ ПОРЕМЕЋАЈА ЗА НИСКЕ САТЕЛИТЕ

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УДК 521.176
Оригинални научни рад

У раду је приказана нова, чисто аналитичка теорија поремећаја кретања за ниске сателите, намењена прелиминарном изучавању њихових путања. Теорија користи мешовит Лагранж-Хамилтонов формализам и засебно третира ефекте врло кратких, средњих и дугих периода. Како ексцентричност путање мора бити мала да би се избегао удар о површину централног тела, теорија је првог степена у односу на исту, док је нагиб путањске равни произвољан. Два главна проблема су спора конвергенција сферних хармоника гравитационог поља централног тела са степеном и редом хармоника и постојање необрачунатих ефеката другог реда. Описана је примена ове теорије у сврху анализе оптималне путање и буџета горива ниског, поларног Месечевог сателита, какав је напр. сателит МОРО, предвиђен у оквиру једне лунарне мисије предложене Европској свемирској агенцији.