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ON THE POTENTIAL ENERGY OF SPHERICALLY SYMMETRIC STELLAR SYSTEMS

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SUMMARY: The case of spherically symmetric, self-consistent, stellar systems, whose total mass is contained within a finite radius, is considered. A general formula for their potential energy, where the latter is expressed in terms of the total-mass square, limiting radius and a dimensionless coefficient depending on the general density slope, is given. Though the increase of this dimensionless coefficient follows the increasing in another dimensionless quantity (ratio limiting radius to half-mass one), the ratio of the two, for the case of the realistic models examined in the present paper, remains almost constant, usually about 0.4-0.5 depending on the given model.

1. INTRODUCTION

As well known, the virial theorem appears as an almost indispensable tool in stellar astronomy, especially for the purpose of mass determination. It is also well known that for self-consistent stellar systems the virial is equal to the potential energy. Such stellar systems are the subject of the present paper.

The most frequent case, when masses are determined (more precisely estimated), is to apply an approximate formula based on the virial theorem where the mass is obtained through the radius and the mean velocity square. The approximative character of such formulae is due to the presence of a dimensionless quantity in the potential energy depending on the system's geometry and slope of the mass distribution, which is usually estimated to the order of magnitude only. Here one should also mention the amount of the radius being an additional problem.

The case of spherical symmetry, as the most simple one, has been, certainly, most frequently treated in the literature. There use is made of some equivalent radii, such as the effective radius (e. g. Kulikovskij, 1985 - p. 216) and the gravitational one (e. g. Binney, Tremaine, 1987 - p. 213), etc. These radii can be related to the physical radii of a spherical stellar system, such as the limiting radius (surrounding the system's total mass) - provided that it is finite, being the case considered in the present paper - and the half-mass one (surrounding half the system's total mass). In the two books mentioned above one can find a relationship between the equivalent radii and the physical ones given for some special cases, but the problem, certainly, deserves more attention. In addition, Binney and Tremaine (1987 - p. 214) mention a result of Spitzer connecting the amounts of the gravitational radius and of the half-mass one (called median by them), however the work of Spitzer (1940) concerns the case of the polytrope model with n=5 (see the explanations below) and it is no result of a very wide application.

2. PROCEDURE

For a self-consistent spherically symmetric stellar system, as can be easily shown, the expression for the potential energy has the following form

$$Wp = -\kappa \frac{GM^2}{r_t} \quad (1)$$

where G is the universal gravitation constant, M the total mass of the system, r_t the limiting radius and $\kappa(> 0)$ – the dimensionless coefficient which, in view of the spherical-symmetry assumption, depends on the slope of the density function only. From (1), bearing in mind the formula given in Binney and Tremaine (1987), it is easy to see that the coefficient κ is, in fact, the ratio of the limiting radius to the gravitational one. In view of the simple relation between the latter and the effective radius (their ratio is 2) which is found from the comparison of the formulae given in the books mentioned above, it is very easy to find that the ratio of the limiting radius to the effective one is equal to 2κ.

This dependence can best be seen through the relationship between κ and another dimensionless coefficient γ which is the ratio of the limiting radius to the half-mass one. As will be seen below any increase in γ is followed by a corresponding increase of κ. In the case of the polytrope with $n = 5$ (n polytrope index) both increase to infinity, but their ratio remains finite, i. e. $\kappa/\gamma = 0.384$ as found by Spitzer (1940). However, for any system contained within infinite limiting radius, but with finite total mass, (like the $n = 5$ polytrope) the ratio κ/γ must be finite in order to obtain a finite potential energy in view of (1).

3. RESULTS

In this paper a few spherically symmetric models where the total mass is contained within a finite radius are examined. These models chiefly contain some parameters so that by varying them one finds the limits of κ for each of them. These limits are given together with those of γ.

The models studied here are the following:

i) the power-law model (Model I) where the mass within an arbitrary radius $M(r)$ depends on r as

$$M(r) \sim r^{\beta}, \qquad 0 \leq \beta \leq 3 \quad (2)$$

ii) the polytrope model (Model II - details e. g. Ogorodnikov, 1958, p. 460);

iii) a model studied by the present author (Model III - formula (1), Ninković, 1991);

iv) another model studied by the same author (Model IV - form. (1), Ninković, 1988);

v) modified Hubble profile (Model V - form. (2-37), Binney and Tremaine, 1987);

vi) King's model (Model VI - form. (27), King, 1962).

The formulae mentioned in the parentheses for the case of Models III-VI describe the density-radius dependence. The formulae describing the density are rather theoretically based than empirically for the case of Models I-III, whereas those appearing in Models IV-VI are purely empirical.

Model I, as easily seen, yields

$$W_p = -\frac{\beta}{2\beta - 1} \frac{GM^2}{r_t}$$

It is clear that for $\beta \leq 1/2$ this model is dynamically unstable, but it is hardly expected to be completely applicable to a realistic stellar system. Due to its simplicity Model I has been used as a first approximation only and generally within spatially limited segments of a stellar system.

As for the other dimensionless coefficient - γ - as easily seen, its value for $\beta = 3$ is $2^{1/3}$; further it increases with κ increasing so that for $\beta = 1/2$, when the latter tends to infinity, γ reaches the value of 4. The ratio κ/γ remains over a significantly wide interval of β almost constant (about 0.5) to begin an abrupt increase as β approaches the critical value of 1/2.

In the case of Model II, as well known (e. g. Ogorodnikov, 1958 - p. 316), the following formula

$$W_p = -\frac{3}{5 - n} \frac{GM^2}{r_t}, \qquad 0 \leq n \leq 5 \quad (3)$$

is valid for the potential energy. Considering that the behaviour of κ is self-evident, one should say that the ratio κ/γ decreases with n increasing, beginning with about 0.48 ($n = 0$) to reach approximate values of 0.45 ($n = 1$), i. e. of 0.38 ($n = 5$) for the three cases allowing an analytic solution. The situation for $n > 5$ corresponds to dynamical instability as evident from (3). In view of the coincidence between Model II ($n = 0$) and Model I ($\beta = 3$) a comparison of them becomes interesting. For example, the case $n = 1$, $\beta = 3/2$ yields the same value for κ - that of 3/4 - and the corresponding values for γ are close to each other - $4^1/3$ (or approximately 1.59, Model I), i. e. about 1.65 (Model II). For higher values of the parameters the difference between the two models becomes more significant.

Unlike the former two Model III has no parameter so that it yields a unique solution $\kappa = 5/7, \gamma = 1.56$. A similar situation appears in the case of Model I with $\beta = 5/3$ where κ has exactly the same value and γ is close to its counterpart ($\gamma \approx 1.516$). However, this similarity is of a global character since in Model III the density has no singularity at the centre and vanishes at the limiting radius. A similar comment is valid for the comparison Model I - Model II.

For the case of Model IV the limits for the two dimensionless coefficients depend on the ratio r_c/r_l (Ninković, 1988 - form. (1)). The extremal cases corresponding to the values of 0 and 1 for this ratio yield: $\kappa = 7/5$, $\gamma = 2.88$, i. e. $\kappa = 0.76$, $\gamma = 1.65$, respectively; the ratio κ/γ remains practically unchanged - it is equal to 0.49 in the former case, i. e. to 0.46 in the latter.

In the case of Model V the potential energy of the system can be obtained only numerically. A solution expressed in terms of (1) depends on the ratio of the limiting radius to the characteristic one (r_c - form. (2-37) of Binney and Tremaine, 1987). An extremal case is when this ratio is equal to 1, but the corresponding opposite case does not exist since then the total mass of the system becomes infinite no matter whether $r_l \to \infty$ or $r_c \to 0$. The results are given in Table 1.

Table 1 The Results for Model V

r_l/r_c	κ	κ/γ
1	0.64	0.47
5	0.93	0.43
10	1.21	0.42
50	2.67	0.44
100	3.9	0.45

In the case of Model VI the situation is more complicated since then even the mass within an arbitrary radius is not obtainable analytically. On the basis of numerical solutions the values for both κ and γ are obtained as functions of the model parameter r_t/r_c; the latter two explained in King (1962). It should be added that in King models the name "tidal radius" is preferred for the limiting radius. The results are presented in Table 2 where for convenience the values for γ are given through the ratio κ/γ as in Table 1.

Table 2 The Results for Model VI

r_t/r_c	κ	κ/γ
1	0.84	0.44
2	0.97	0.44
5	1.34	0.42
10	1.81	0.41
20	2.53	0.40
50	4.07	0.40
100	6.10	0.42
200	9.38	0.45

4. DISCUSSION AND CONCLUSIONS

For a special case - a self-consistent stellar system with spherical symmetry represented by a model with finite limiting radius - the general formula yielding the potential energy (1) is derived. The dimensionless coefficient - κ - in this formula is related to the general density slope in the system. Its lowest value, that of 3/5, corresponds to the homogeneous sphere; the more significant is the decreasing in the density, the higher value is acquired by κ. The best illustration of this dependence is through the relationship between κ and γ (the ratio of the limiting radius to that of half mass) where the increasing in one of them is followed by the corresponding increase of the other one. This circumstance enables rewriting of (1) to be made in such a way that κ is replaced by the ratio κ/γ and instead of the limiting radius there appears the half-mass one. With the exception of Model I which, though very simple, is nevertheless, sufficiently unrealistic, all other models examined here prefer a rather constant value of κ/γ ratio, about 0.4-0.5, which might be a general characteristic of realistic models of self-consistent stellar systems with spherical symmetry.

Model IV and Model VI, among the ones examined in the present paper, deserve a special attention. They are empirical and they are applied to coronae of spiral galaxies, i. e. to globular clusters and dwarf galaxies, respectively, which are the stellar systems sufficiently close to the self-consistency condition, unlike halos and bulges of spiral galaxies to which Model V is applicable.

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REFERENCES

Binney, J., Tremaine, S.: 1987, *Galactic Dynamics*, Princeton University Press, Princeton, New Jersey.

King, I.: 1962, *Astron. J.*, 67, 471.

Kulikovskij, P. G.: 1985, *Zvezdnaya astronomiya*, Nauka, Glav. red. fiz. - mat. lit., Moskva.

Ninković, S.: 1988, *Bull. Obs. Astron. Belgrade*, 139, 21.

Ninković, S.: 1991, *Bull. Obs. Astron. Belgrade*, 144, 1.

Ogorodnikov, K. F.: 1958, *Dinamika zvezdnykh sistem*, Gos. izdat. fiz. - mat. lit., Moskva.

Spitzer, L. Jr.: 1940, *Mon. Not. R. Astr. Soc.*, 100, 396.

О ПОТЕНЦИЈАЛНОЈ ЕНЕРГИЈИ СФЕРНО СИМЕТРИЧНИХ ЗВЕЗДАНИХ СИСТЕМА

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Разматра се случај само-усаглашених сферно симетричних звезданих система чија је укупна маса садржана унутар коначног полупречника. Дата је општа формула за њихову потенцијалну енергију где се ова величина изражава преко квадрата укупне масе, граничног полупречника и једног бездимензионог коефицијента чија вредност зависи од просечног градијента густине. Премда раст овог коефицијента следи рашћење једног другог бездимензионог коефицијента (односа између граничног и тзв. "полупречника пола масе"), однос ова два, за случај реалистичних модела испитаних у овом раду, остаје скоро константан, обично око 0,4–0,5 у зависности од датог модела.